

Competitive Equilibria with Indivisible Goods and Generic Budgets

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We study the existence and fairness properties of competitive equilibria in the fundamental Fisher market model, in which players have budgets of artificial currency (“money”), which they do not value. In our model, the standard assumption of divisible goods is replaced with an assumption of *indivisibility*. A competitive equilibrium fails to exist in one of the simplest possible such markets: two players with *equal* budgets and a single indivisible good. However, this turns out to be a knife-edge instance – an equilibrium does exist once budgets are made *generic* by slight perturbation. Is non-existence of an equilibrium a knife-edge phenomenon more generally? I.e., do generic budgets guarantee equilibrium existence in more complex markets, with multiple different goods, more general player preferences, or far-from-equal budgets? Focusing on the class of additive preferences, we prove several existence results for two players with generic budgets. On the flip side, we demonstrate non-existence for general preferences, despite generic budgets. We also study fairness properties of competitive equilibria, suggesting new notions of fair allocation among players with *different entitlements* to the goods being allocated.

1 INTRODUCTION.

We study competitive equilibria in markets of indivisible goods without money, where players have generic and possibly very different artificial budgets. Such markets capture real-life applications such as allocating courses to students or shifts to workers, or sharing scientific/computational resources within a university or company.

1.1 Motivation.

Our motivation is two-fold: first, we wish to identify conditions for the existence of a competitive equilibrium in markets of indivisible goods; second, we wish to develop notions of fairness that apply to the allocation of such goods among players with different entitlements.

1.1.1 Fisher markets and equilibrium with indivisible goods. A fundamental achievement of general equilibrium theory is establishing the existence of competitive market equilibrium in “convex” settings [5]. In a competitive equilibrium, goods are assigned prices, each player takes his preferred set of goods among those that are within his budget, and the market clears. By the first welfare theorem, the resulting allocation is Pareto efficient. However, when there are indivisible goods, equilibrium existence is no longer guaranteed.

In this paper we focus on the simple Fisher market model [11], where a seller brings m goods to the market and the n players (buyers) bring certain positive amounts of artificial currency in the form of budgets b_1, \dots, b_n . The seller has no use for the goods and the players have no use for their budget, but each player has a preference order among all possible bundles of goods.

If some goods are divisible, a competitive equilibrium is known to exist when players’ preferences satisfy mild conditions (e.g., [29]). We focus on the case where *all* goods are indivisible. It is not hard to see that an equilibrium need not exist – just consider a single indivisible item sold among two players, both with the same budget of 1. Indeed, if the item’s price is at most 1 then both players desire it, while if the price is strictly above 1 then neither can afford it and the market does not clear.¹

¹Recall that players have no value for money, so a player is not satisfied if the price is 1 but he does not get the item.

This simple non-existence example turns out to be a knife-edge phenomenon: if the budgets are $1 + \epsilon$ and 1 instead of precisely equal, then an equilibrium exists by setting the item price in between the two budgets. To avoid such knife-edge non-existence results, Budish [18] initiates the study of *almost-equal* budgets.² In this paper we further advance this idea by considering *generic* budgets – arbitrary budgets (possibly far from equal) to which small perturbations have been added.³

While generic budgets may not be a silver bullet solution,⁴ anecdotal evidence gathered from computer simulations and real-life data suggests that existence in our model is quite common with generic budgets. A computerized search we ran over small (2-player) markets with generic budgets produced no non-existence examples; similarly, a simple tâtonnement process always found an equilibrium in larger markets with real-life preferences taken from the *Spliddit* website [35] (details appear in Appendix E). *This motivates seeking a finer understanding of conditions under which equilibrium existence is guaranteed in Fisher markets with indivisible goods and generic budgets.*

1.1.2 Fairness with indivisible goods and different entitlements. Which allocations of goods among players should be considered “fair”? A vast literature is devoted to this question (for expositions see, e.g., [12, 14, Chapters 11–13]). Many standard notions of fairness in the literature reflect an underlying assumption that all players have an equally-strong claim to the goods. However, some players may be *a priori* much more entitled to the goods than others (see, e.g. [51, Chapters 3 and 11]). Real-life examples include partners who own different shares of the partnership’s holdings [24], different departments sharing a company’s computational resources [34], or family members splitting an heirloom [49]. Our model captures these situations: the entitlement of player i is given by his budget b_i . We may assume without loss of generality that $\sum_i b_i = 1$.

We seek an appropriate notion of fairness that takes into account the indivisible goods and different entitlements, and captures the idea of “proportional satisfaction of claims” [17, p. 95]. In the special case where all entitlements are equal ($b_1 = \dots = b_n = 1/n$), *envy-freeness* (no player prefers another’s allocation to his own) is an important fairness criterion [31], but it is not clear how to generalize this notion to heterogeneous entitlements. *Fair share* (every player i prefers his allocation to a b_i -fraction of all items) is also an important fairness criterion [56], but it does not easily extend to indivisible items.

In this paper we adapt a well-known approach to fair allocation with equal entitlements – finding a *competitive equilibrium from equal incomes (CEEI)* [8, 59] – to the case of unequal entitlements. We treat the players as buyers and their entitlements as budgets, and seek a competitive equilibrium in the resulting Fisher market. Such an equilibrium is not guaranteed to exist, but this is to be expected – indivisibility can indeed undermine the ability to fairly allocate. Our idea is to first perturb the entitlements (budgets), since competitive equilibrium from *generic* budgets may exist widely and offer approximate fairness guarantees that are the best possible given indivisibilities. *Thus, a better understanding of competitive equilibrium with generic budgets can help reach fair or approximately fair division of indivisible items among players with heterogeneous entitlements.*

²Budish [18] couples almost-equal budgets with an *approximate* equilibrium notion that requires giving up market clearance; in contrast, our results will not require market clearance relaxation.

³A more formal way to define generic budgets is as budgets that do not satisfy a few simple equalities, such as $b_1 = b_2$.

⁴An example from a different model is the following: In the *quasi-linear* model, alongside the indivisible goods there is a single infinitely divisible item that plays the role of money. When budgets are sufficiently large that they are effectively unlimited, this coincides with the combinatorial auction setting, in which market equilibrium existence is guaranteed only for gross substitutes preferences [36].

1.2 Our results.

1.2.1 The setup. Consider a market setting with n players, in which each player i has an *ordinal* preference $<_i$ over all subsets of the m heterogeneous items.⁵ Unless stated otherwise we assume throughout that $<_i$ is *strict* ($S <_i T \implies T \not<_i S$ for every two item sets $S \neq T$), and *monotone* ($S <_i T$ for every $S \subset T$).⁶ For example, if we are allocating 3 items $\{A, B, C\}$, a possible preference of player i might be $\emptyset <_i \{A\} <_i \{B\} <_i \{A, B\} <_i \{C\} <_i \{B, C\} <_i \{A, C\} <_i \{A, B, C\}$. Each player has a budget $b_i > 0$, and without loss of generality $\sum_i b_i = 1$.

A *competitive equilibrium (CE)* consists of a pair: a price vector $p = (p_1, \dots, p_m)$, and an allocation (partition) $(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n)$ of all the items ($\mathcal{S}_i \cap \mathcal{S}_k = \emptyset$ for $i \neq k$ and $\bigcup_i \mathcal{S}_i = \{1, 2, \dots, m\}$). Every player i is allocated his most preferred set among all sets of which the price is within his budget: $\sum_{j \in \mathcal{S}_i} p_j \leq b_i$, and $\sum_{j \in T} p_j > b_i$ for every $T >_i \mathcal{S}_i$.

1.2.2 General ordinal preferences. Our results on the existence of a CE with general preferences and generic budgets can be summarized by the following theorem. The theorem rules out a “sweeping” existence result that holds for every single market, but explains why non-existence will not be apparent in very small markets (up to 4 items). Theorem 1.1 motivates our focus on non-general classes of preferences in the next sections.

THEOREM 1.1 (GENERAL PREFERENCES). *There exists a market with 5 items and 2 players such that for an open interval of budgets, no CE exists. On the other hand, a CE with generic budgets always exists in markets with:*

- At most 3 items and any number of players.
- 4 items and 2 players.

The existence results for small markets is surprising in comparison to quasi-linear settings, in which a CE may fail to exist even when there are only 2 indivisible items and 2 players.

Our theorem leaves open the case of 4 items and more than 2 players. In subsequent work, Segal-Halevi [53] settles this open question, showing guaranteed existence for 4 items and 3 players, and demonstrating possible non-existence for 4 items and 4 players, even with additive valuations.

When a CE does exist, we show that the CE allocation has a natural fairness property related to the well-known cut-and-choose protocol. We briefly explain this property here, and defer the details to Section 3.1. Consider 2 players, and recall that in the cut-and-choose protocol one player (the “cutter”) divides the items into 2 sets, and the other player chooses his favorite set; the resulting allocation is intuitively fair. Budish [18] defines an allocation as *MMS fair* if each player gets a bundle she prefers at least as much as the one she can guarantee for herself as the cutter, called her *maximin share (MMS)*. We show that CE allocations have a generalized MMS fairness property appropriate for the case of different budgets. Define an ℓ -out-of- d *maximin share* of a player to be a bundle that is at least as good as the one he can guarantee for himself by the following protocol: the player partitions the items into d (possibly empty) parts, and then takes the worst ℓ of these (because other players get to choose $d - \ell$ parts first, and their choice is assumed to be worst-case for the divider).

THEOREM 1.2 (FAIRNESS). *Consider a CE allocation in a market with n players. For every $i \in [n]$ and rational number $\ell/d \leq b_i$, in the CE allocation player i gets an ℓ -out-of- d maximin share.*

⁵ $<_i$ is a linear order, i.e., transitive, antisymmetric and total.

⁶ Monotonicity is a natural assumption in most applications [10, 39, Sec. 12.1.4]. For one exception see [18] – enrolling in all courses can be less preferable for a student to enrolling in a subset.

1.2.3 Main existence results: 2 players and additive preferences. At this point we turn our attention to the simplest case in which it is not known whether generic budgets guarantee the existence of a CE – the case of additive preferences and 2 players. The players have item values $v^1, \dots, v^m > 0$, and $S < T$ if and only if $\sum_{j \in S} v^j < \sum_{j \in T} v^j$. Note that the following results hold even when there are multiple copies of the same items (in which case preferences are no longer strict). Our first result is establishing the second welfare theorem (the first welfare theorem is known to hold):

PROPOSITION 1.3 (SECOND WELFARE THEOREM). *Consider 2 players with additive preferences. For every Pareto efficient allocation $S = (S_1, S_2)$, there exist budgets b_1, b_2 and prices p such that (S, p) is a CE.*

We show that the second welfare theorem does not hold for general preferences – there exists a 2-player market with non-additive preferences in which a certain Pareto efficient allocation cannot be supported by any budgets and prices.

As our main result, we identify large families of instances for which a CE exists:

THEOREM 1.4 (ADDITIVE PREFERENCES). *Consider 2 players with additive preferences v_1, v_2 . A CE with generic budgets always exists in markets satisfying any one of the following conditions:*

- *Existence of a proportional allocation $S = (S_1, S_2)$ in which every player i receives at least a b_i -fraction of his total value.*
- *Budgets are almost equal, as in Budish’s setting (but without revoking market clearance).*
- *The 2 players have identical preferences.*

Moreover, the CEs guaranteed to exist assign identical prices to identical copies of the same item.

Our results for 2 additive players are via a characterization we develop of CEs for 2 players, which has a particularly nice form for additive preferences. We use this to show that linear combinations of the additive preferences form equilibrium prices under a certain condition. To see when this condition is fulfilled, we use a graphical representation of all allocations and their values for the players. Details appear in Section 4.

Just like for general preferences, when a CE exists for additive preferences we are interested in its fairness properties. The first condition of Theorem 1.4 shows that CEs are no rarer than allocations which satisfy the fairness property of proportionality. As for the second condition, we know from [18] that a CE with almost equal budgets satisfies “envy-freeness up to one item”, so if the goal is to reach such envy-freeness then it is always achievable for 2 additive players. The third condition establishes CE existence for the case that would seem to be hardest in terms of fairness, in which players are in direct competition. We also show that in all cases to which Theorem 1.4 applies, the CE has the following fairness property: each player i receives his *truncated share*, which is as close to proportionality as the indivisibility of the items allows when no proportional allocation exists.⁷

We leave the following as our main open problem:

OPEN PROBLEM 1.5. *Does there always exist a CE for 2 players with additive preferences and generic budgets?*

1.2.4 Relevant classes of preferences. In light of the non-existence result for general preferences, we consider the following hierarchy of preference classes, which we prove in Appendix A:

$$\text{LEXICOGRAPHIC} \subsetneq \text{ADDITIVE} \subsetneq \text{RESPONSIVE} \subsetneq \text{GENERAL}.$$

⁷We define truncated share as any bundle the player prefers at least as much as his most preferred Pareto efficient allocation $S = (S_1, S_2)$ for which $v_i(S_i) \leq b_i \cdot v_i(\{1, \dots, m\})$.

One may hope to get a finer hierarchy from preferences represented by valuations from the “subadditive hierarchy” in quasi-linear markets [38], but we show that already the class of submodular valuations can represent arbitrary preferences:

PROPOSITION 1.6 (SUBMODULAR REPRESENTATION). *For every strict and monotone ordinal preference “ $<$ ”, there exists a submodular valuation v such that $S < T$ if and only if $v(S) < v(T)$. I.e.,*

$$\text{SUBMODULAR} = \text{GENERAL}.$$

We also consider an incomparable class of preferences that we call *leveled*: a player has a leveled preference if he prefers sets with larger cardinality, i.e., $|S| < |T|$ implies $S < T$ (preferences among sets of the same cardinality can be arbitrary).

PROPOSITION 1.7 (EXISTENCE FOR CERTAIN PREFERENCE CLASSES). *A CE with generic budgets always exists in markets with:*

- Any number of players and lexicographic preferences.
- 2 players and leveled preferences.

1.2.5 Organization. After presenting the preliminaries (Section 2), Section 3 includes our results for general ordinal preferences, and Section 4 covers cardinal additive preferences (including our main technical result). In Section 5 we conclude and present some open problems.

Due to space considerations, some of our results and many of the proofs are deferred to the appendices. Our study of subclasses of ordinal preferences is presented in Appendix A. In Appendix B we include detailed background on fairness and how our new fairness notions fit into the literature. Missing proofs appear in Appendix C (including our main technical proof in Appendix C.3). Appendix D discusses computational complexity issues and Appendix E summarizes our computer simulations.

1.3 Additional related work.

Discrete Fisher markets. Our model is a special case of Arrow–Debreu exchange economies with indivisible items. Several other variants have been studied: [2, 32, 43, 45, 57] consider markets with indivisibilities in which an infinitely divisible good plays the role of money, and so money carries inherent value for the players. Shapley and Scarf [54], Svensson [58] and subsequent works focus on the house allocation problem with *unit-demand* players. Several works assume a continuum of players [e.g., 41], and/or study relaxed CE notions [e.g., 26, 28, 50]. Closest are the models of *combinatorial assignment* [18], which allows non-monotonic preferences, and of *linear markets* [26], which crucially allows non-generic budgets (for further discussion of [26] see Appendix D.1).

CEEI. Budish [18] circumvents the non-existence of CEEI due to indivisibilities by weakening the equilibrium concept and allowing market clearance to hold only approximately. He focuses exclusively on budgets that are almost equal. In the same model, Othman et al. [47] show PPA-completeness of computing an approximate CEEI, and NP-completeness of deciding the existence of an approximate CEEI with better approximation factors than those shown to exist by Budish. The preferences used in the hardness proofs are non-monotone, leading Othman et al. to suggest the research direction of restricting the preferences (as we do here) as a way around their negative results. Brânzei et al. [15] study (exact) CEEI existence for two valuation classes (perfect substitutes and complements) with non-generic budgets. For *divisible* items, there has been renewed interest in CEEI under additive preferences due to their succinctness and practicality. Bogomolnaia and Moulin [8] offer a characterization based on natural axioms, and Bogomolnaia et al. [9] analyze CEEI allocations of “bads” rather than goods.

Fairness with indivisibilities. Most notions of fair allocation in the classic literature apply to divisible items. Bouveret et al. [10] survey research on fairness with *indivisible* items, emphasizing computational challenges (see also [6, 25]). The works of [39, 40] study envy minimization (rather than elimination) with indivisibilities. Guruswami et al. [37] study envy-freeness in the context of profit-maximizing pricing. Several recent works [4, 16, 20, 22, 23, 33] consider the approximation of Nash social welfare and related fairness guarantees. As for practical implementations of fair division with indivisibilities, these are discussed by Budish and Cantillon [19] (for responsive preferences), Othman et al. [46] (implementing findings of Budish [18]), Goldman and Procaccia [35] (presenting the Spliddit website for additive preferences), and Brams and Taylor [12] (presenting the adjusted winner algorithm in which one item may need to be divided). Mechanism design aspects appear in [3]. For further discussion see Section 2.2.

Concurrent and subsequent work. Continuing the literature above, there has been a recent wave of works on fair division of indivisible items, appearing concurrently or subsequently to early versions of our work. Farhadi et al. [30] independently develop a new fairness notion for allocation among agents with cardinal preferences and different entitlements. Their notion is distinct from those that we suggest and is not directly related to the solution concept of a CE. For example, their fairness notion is not always guaranteed when allocating 3 items among players with generic budgets (as implied in [30, Theorem 2.1]). This is at odds with our CE existence result for 3 items and generic budgets (Proposition 3.3), which guarantees fairness according to our notions (Proposition 3.2), demonstrating the difference between the approaches. [48] recently study the envy-free relaxation $EF-1^*$ suggested by [20]. The works of [7] and [21] study envy-freeness with incomplete knowledge.

2 PRELIMINARIES.

2.1 Market preliminaries.

Market setting. A (*discrete Fisher*) market consists of m indivisible items M and n players N . Each player $i \in N$ has an *ordinal* preference $<_i$ among subsets of items, which we refer to as bundles and often denote by S or T . We assume preferences are *monotone*, and unless stated otherwise, *strict*: preference $<_i$ is monotone (satisfies free disposal) if $S <_i T$ whenever $S \subset T$, and strict if there are no indifferences among bundles.

Some of our results apply to classes of *cardinal* preferences, i.e., valuation functions which assign to every bundle S a nonnegative value $v(S)$. Here cardinality allows to compare *how much* a bundle S is preferred *relative to* another bundle T .⁸ In particular, we study the class of cardinal additive preferences, where v is *additive* if $v(S) = \sum_{j \in S} v(\{j\})$ for every bundle S . Additive preferences can be non-strict; here we assume strictness unless stated otherwise. We remark that any ordinal preference $>$ can be *represented* by some (cardinal) valuation, where a preference $>$ is represented by a valuation v if for every two bundles S, T , $S > T \iff v(S) > v(T)$.⁹

In addition to preferences, players in our model have *budgets*. Let $b = (b_1, b_2, \dots, b_n)$ be a budget profile, where $b_i > 0$ is player i 's budget. Unless stated otherwise, we assume without loss of generality (wlog) that $\sum_{i=1}^n b_i = 1$ and $b_1 \geq b_2 \geq \dots \geq b_n$ (every market can be converted to satisfy these properties by renaming and normalization). We emphasize that budgets and “money” in our model are means for allocating items among players with different *a priori* entitlements; in particular, money has no intrinsic value for the players, whose preferences are over subsets of items only (disregarding any remaining budget).

⁸The absolute values do not matter; hence a cardinal valuation can be normalized without loss of generality.

⁹In a valuation representing an ordinal preference, even the relative values do not matter beyond the order they induce over bundles.

Market outcome. An allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n)$ is a partition of *all* items among the players (i.e., $\mathcal{S}_i \cap \mathcal{S}_k = \emptyset$ for $i \neq k$ and $\bigcup_i \mathcal{S}_i = M$). By definition, an allocation is *feasible* (no item is allocated more than once) and *market clearing* (every item is allocated).¹⁰ Let $p = (p_1, p_2, \dots, p_m)$ be non-negative item prices. The price $p(\mathcal{S})$ of a bundle is $\sum_{j \in \mathcal{S}} p_j$. We say that \mathcal{S} is *within budget* b_i if $p(\mathcal{S}) \leq b_i$, and that it is *demanded* by player i at prices p if it is the most preferred bundle within his budget (i.e., $p(\mathcal{S}_i) \leq b_i$ and $p(T) > b_i$ for every $T \succ_i \mathcal{S}_i$).

Definition 2.1. Consider a market with budget profile b . A *competitive equilibrium (CE)* is a pair (\mathcal{S}, p) of allocation \mathcal{S} and item prices p , such that \mathcal{S}_i is demanded by player i at prices p for every i .

Definition 2.2. Consider player preferences $\{\succ_i\}_{i \in N}$. An allocation \mathcal{S} is *Pareto optimal (PO)* (a.k.a. Pareto efficient) if no allocation \mathcal{S}' dominates it, i.e., for every $\mathcal{S}' \neq \mathcal{S}$ there is some player i for whom $\mathcal{S}_i \succ_i \mathcal{S}'_i$.

THEOREM 2.3 (FIRST WELFARE THEOREM). *Let (\mathcal{S}, p) be a CE. Then \mathcal{S} is PO.*

A proof appears in Appendix C for completeness.

Supported allocations and budget-exhausting prices. Given a market with budget profile b , an allocation \mathcal{S} is *supported* in a CE if there exist item prices p such that (\mathcal{S}, p) is a CE. Where only preferences are given, we overload this notion and say that \mathcal{S} is supported in a CE if there exist prices p and budgets b such that (\mathcal{S}, p) is a CE in a market with the given preferences and budgets b . Given budgets b and an allocation \mathcal{S} , we say that prices p are *budget-exhausting* if $p(\mathcal{S}_i) = b_i$ for every player i . Note that if \mathcal{S} is budget-exhausting then $\mathcal{S}_i \neq \emptyset$ as $b_i > 0$. We observe that budget-exhaustion is wlog when every agent is allocated, and use this observation throughout the paper.

CLAIM 2.4. *For every CE (\mathcal{S}, p) such that $\mathcal{S}_i \neq \emptyset$ for every i , there exists a CE (\mathcal{S}, p') in which p' is budget-exhausting.*

PROOF. As every player is allocated at least one item, we can raise the price of that item in his allocated bundle until his budget is exhausted. The new prices form a CE with the original allocation since every player still gets his demanded set. \square

2.2 Fairness preliminaries.

We include here the fairness preliminaries most related to our work. A more detailed account appears in Appendix 2.2 and is summarized in Tables 1-2, which also show where our new fairness notions fit in with existing ones. The relation to notions such as Nash social welfare and EF-1* is discussed in the appendix.

In general, fairness notions can be categorized by whether they are defined for ordinal or cardinal preferences, divisible or indivisible items, and unweighted versus budgeted players. Our main concern is with fairness notions for indivisible items and budgeted players, for which not much is known (neither for ordinal nor cardinal preferences).

The “two most important tests of equity” according to Moulin [44, p.166] are (i) guaranteeing each player his fair share; and (ii) envy-freeness. Our main focus will be on fair share, “probably the least controversial fairness requirement in the literature” [9].

2.2.1 Ordinal preferences. Fair share and envy-freeness are well-defined for ordinal preferences when items are *divisible*. Intuitively, fair share guarantees that each player believes he receives at least $1/n$ of the “cake” being divided, and envy-freeness guarantees he believes no one else receives

¹⁰With strict preferences, market clearance is necessary for Pareto efficiency.

a better slice than him. More formally, *fair share* (FS) requires for each player to receive a bundle that he prefers at least as much as the bundle consisting of a $1/n$ -fraction of every item on the market. Player i *envies* player k given allocation \mathcal{S} if $S_i <_i S_k$, and an allocation is *envy-free* (EF) if no player envies another player.

When items are indivisible, FS is not well-defined; EF is well-defined but often cannot be satisfied [27]. To circumvent the definition and existence issues stemming from indivisibilities, Budish [18] proposes appropriate variants:

Definition 2.5 (FS with indivisibilities [18]). An allocation \mathcal{S} guarantees *1-out-of- n maximin share* (a.k.a. has the MMS property) if every player receives a bundle he prefers at least as much as the bundle he can guarantee in the worst case by the following procedure: partitioning the items into n parts, and allowing $n - 1$ players to choose their parts first (worst case is over the choice of the other players).

Definition 2.6 (EF with indivisibilities [18]). An allocation \mathcal{S} is *envy-free up to one good* (EF-1) if for every two players i, k , for *some* item $j \in S_k$ it holds that $S_k \setminus \{j\} <_i S_i$.

We discuss how to generalize the maximin share guarantee for budgeted players in Section 3.1, defining the notion of ℓ -out-of- d maximin share (Definition 2.5). We also define a notion called justified-EF for budgets (Definition B.7), and leave the question of how to generalize (non-justified) EF to budgets as an open direction.

Relation to CE. It is well-known that every CE with n equal budgets gives every player his 1-out-of- n maximin share and achieves EF. Budish [18] shows that every CE with almost equal budgets guarantees 1-out-of- $(n + 1)$ maximin share (whereas 1-out-of- n maximin share cannot always be guaranteed). Proposition 3.2 generalizes this to arbitrary budgets using the notion of ℓ -out-of- d maximin share.

2.2.2 Cardinal preferences. For cardinal preferences, the parallel of FS is the notion of proportionality, which extends naturally to players with different budgets: Given a budget profile b , an allocation \mathcal{S} gives player i his *budget-proportional share* if player i receives at least a b_i -fraction of his value for all items, that is $v_i(S_i) \geq b_i \cdot v_i(M)$. An allocation is *budget-proportional* (a.k.a. weighted-proportional) if every player receives his proportional share. When all budgets are equal, such an allocation is simply called *proportional*.

Budget-proportional allocations play a central role in our positive results in Section 4: While it is clear that a budget-proportional allocation does not always exist with indivisible items (e.g., with a single one), we present a relaxation of budget-proportional share that we call truncated share (Definition 4.7). In all settings for which we prove CE existence in this section, the CE guarantees every player his truncated share.

3 GENERAL ORDINAL PREFERENCES.

In this section we study CE existence and fairness for general ordinal preferences. In Section 3.1 we show a fairness property guaranteed by every CE. In Section 3.2 we show existence of a CE for ≤ 3 items and n players with generic budgets, and *non*-existence of a CE for 5 items, 2 players and an open interval of budgets. We also rule out a second welfare theorem for general preferences. The case of 4 items is discussed in Appendix C.

3.1 Fairness of CE with arbitrary budgets.

We define a generalized version of the 1-out-of- n maximin share guarantee (Definition 2.5) that is intended to accommodate arbitrary, possibly very different budgets. Informally, player i 's ℓ -out-of- d maximin bundle is the bundle he can guarantee for himself by the following protocol: the player

partitions the items into d parts, lets the other players choose $d - \ell$ of these parts (their choice is assumed to be worst-case), then receives the remaining ℓ parts. For example, an additive player who values items (A, B, C) at $(1, 2, 3)$ has a 2-out-of-3 maximin bundle of $\{A, B\}$. More formally:

Definition 3.1. The ℓ -out-of- d maximin bundle of player i is the bundle

$$\max_{<i} \left\{ \min_{<i} \left\{ \bigcup_{t \in L} T_t \mid (T_1, \dots, T_d) \text{ is a partition of the items into } d \text{ parts, } L \subseteq [d], |L| = \ell \right\} \right\},$$

where the maximum and minimum are with respect to i 's preferences. An allocation \mathcal{S} guarantees player i his ℓ -out-of- d maximin share if \mathcal{S}_i is either his ℓ -out-of- d maximin bundle or any bundle he prefers to it.

PROPOSITION 3.2. *Let b be an arbitrary budget profile. Every CE guarantees player i his ℓ -out-of- d maximin share for every rational number $\ell/d \leq b_i$.*

Before proving Proposition 3.2 we briefly demonstrate possible applications.

- Consider 2 players with different budgets that are not almost equal, say $b_1 = 2/3, b_2 = 1/3$. The proposition implies that every CE allocation gives player 1 his 2-out-of-3 share, and player 2 his 1-out-of-3 share. Note that for the proposition to hold, there is no need to restrict the number of items m to be ≥ 3 . However, if there are only ≤ 2 items, player 2's 1-out-of-3 maximin bundle is empty, and so the guarantee of 1-out-of-3 share is weak in this case (this is inevitable given the indivisible nature of the items).
- Consider 3 items and a player with budget $5/13$. Allowing for any rational number $\ell/d \leq 5/13$ strengthens the fairness guarantee: Dividing the 3 items into 13 bundles and taking the worst 5 bundles does not guarantee the player anything beyond an empty bundle, whereas a 1-out-of-3 maximin share (using that $1/3 \leq 5/13$) guarantees the player at least one item.
- Proposition 3.2 subsumes a result of Budish [18], who shows that for every CE with equal budgets the equilibrium allocation gives every player his 1-out-of- n maximin share, and if the budgets are almost equal then every player gets his 1-out-of- $(n + 1)$ maximin share. Indeed, if budgets are equal, then $b_i \geq 1/n$ for every player i . If budgets are almost equal, then $b_1 \geq b_2 \geq \dots \geq b_n \geq \frac{n}{n+1} b_1 \geq \frac{1}{n+1}$ for every player i (the last inequality follows since b_1 must be $\geq 1/n$ for the budgets to sum up to 1). Thus the result of Budish can be deduced from Proposition 3.2.

PROOF OF PROPOSITION 3.2. Let (S, p) be a CE. Since S is an allocation of all items, every item is bought by an agent and so $P := \sum_j p_j \leq \sum_i b_i = 1$. Let (T_1, \dots, T_d) be any partition of the items into d parts, and observe that $1 \geq P = \sum_j p_j = p(\bigcup_{t=1}^d T_t) = \sum_{t=1}^d p(T_t)$. By the pigeonhole principle, there exists a set of ℓ parts whose total price is at most $\frac{\ell}{d} P \leq \frac{\ell}{d}$. Let us call this "the cheap subset". By the assumption in Proposition 3.2, $\frac{\ell}{d} \leq b_i$. Therefore, agent i can afford the cheap subset, and by definition of CE, the bundle actually allocated to agent i must be at least as preferred by him as the cheap subset. \square

3.2 Existence and non-existence of CE.

For 2 items and n players whose two highest budgets are distinct, the following is a CE: let player 1 pay b_1 (the highest budget) for his most preferred item, and let player 2 pay b_2 (the second-highest budget) for the remaining item. In Appendix C, we verify by case analysis that a CE always exists for 3 items under a similar genericity condition on the budgets:

PROPOSITION 3.3 (3 ITEMS). *Consider 3 items and $n \geq 2$ players with monotone strict preferences and budget profile b . If $b_1 > b_2 > b_3 \geq 0$ (where $b_3 = 0$ if $n = 2$) then a CE exists.*

We next show that with 5 items and even 2 players, a CE may not exist even for an open interval of budgets, and moreover the second welfare theorem fails to hold. Our negative results follow from Example 3.4:

Example 3.4 (2-player 5-item market). We refer to player 1 as Alice and player 2 as Bob, and denote the items by A, B, C, D, E . The ordinal preferences are induced by the following cardinal values: For the first two items, Alice and Bob have different perspectives. Alice values item A at 10 and item B at 20, and the pair at 700. Bob has a value of 500 for A and 501 for B , and a value of 502 for both. Both Alice and Bob have additive preferences over the three items C, D and E , valuing them at 201, 202 and 203, respectively. For both players, the value of a set is additive across the pair $\{A, B\}$ and the three remaining items; for example, if Alice gets $\{A, B, C, D\}$ her value is $700 + 201 + 202 = 1103$.

Lemma 3.5 and Proposition 3.6 analyze Example 3.4; their proofs appear in Appendix C.

LEMMA 3.5. *In Example 3.4, consider any allocation S where Alice gets $X_1 \in \{A, B\}$ and $Y_1, Y_2 \in \{C, D, E\}$, $Y_1 \neq Y_2$, and Bob gets the remaining items $X_2 \in \{A, B\}$ and $Y_3 \in \{C, D, E\}$. Then S is PO, and is not supported in a CE for any budgets b_1, b_2 .*

PROPOSITION 3.6 (5 ITEMS). *There exist monotone strict preferences for 2 players over 5 items, such that for any budgets $(4/3)b_2 > b_1 > b_2$, no CE exists.*

The next corollary follows immediately from Lemma 3.5:

COROLLARY 3.7 (NO SECOND WELFARE THEOREM). *There exist monotone strict preferences for 2 players over 5 items, and a PO allocation S , such that S is not supported in a CE for any pair of budgets.*

4 TWO PLAYERS WITH CARDINAL ADDITIVE PREFERENCES.

In this section we focus on 2 players with additive preferences and possibly very different budgets. In Section 4.1 we present a characterization of CE for 2 players with any cardinal preferences. In Section 4.2 we use our characterization to prove a version of the second welfare theorem for additive preferences. Our main technical contribution is Section 4.3, where we present several natural conditions, each of them sufficient to guarantee existence of a CE. The main technical result in this section is Theorem 4.10, the proof of which we illustrate via Figures 3 to 5.

Throughout this section we assume wlog that v is normalized, i.e.,

$$v(M) = 1.$$

Weak additive preferences. We remark that our results in this section hold even if we allow items to be identical, and the preferences to be weakly-monotone rather than strictly so. All the CEs in our existence results assign such items identical prices.¹¹

4.1 A characterization of CE for 2 players with cardinal preferences.

We present necessary and sufficient conditions for a budget-exhausting pricing and PO allocation to form a CE, when the preferences of two players are cardinal, and each player gets a non-empty set (in this case budget-exhaustion is wlog by Claim 2.4). We use the following standard notation: For a preference v and disjoint sets X, Y , the *marginal value* of X given Y is denoted by $v_i(X | Y) = v_i(X \cup Y) - v_i(Y)$.

¹¹In [18], the model inspired by course allocation allows for multiple identical units (seats) of each item (classes). Prices are assigned to items rather than to individual units, so that players receiving identical units pay the same. We achieve the same effect.

PROPOSITION 4.1 (CHARACTERIZATION). *Given 2 players with weakly-monotone cardinal preferences v_1, v_2 , consider a PO allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ in which $\mathcal{S}_i \neq \emptyset$ for $i \in \{1, 2\}$, and budget-exhausting item prices p . Then (\mathcal{S}, p) forms a CE if and only if for $i, k \in \{1, 2\}, i \neq k$, and for every two bundles $S \subseteq \mathcal{S}_i, T \subseteq \mathcal{S}_k$,*

$$v_i(S \mid \mathcal{S}_i \setminus S) > v_i(T \mid \mathcal{S}_i \setminus S) \text{ and } v_k(S \mid \mathcal{S}_k \setminus T) > v_k(T \mid \mathcal{S}_k \setminus T) \implies p(S) > p(T). \quad (1)$$

PROOF. For the first direction, assume by way of contradiction that (\mathcal{S}, p) is a CE but Condition (1) is violated. Wlog assume that this is the case for $S \subseteq \mathcal{S}_1$ and $T \subseteq \mathcal{S}_2$, i.e., it holds that $v_1(S \mid \mathcal{S}_1 \setminus S) > v_1(T \mid \mathcal{S}_1 \setminus S)$ and $v_2(S \mid \mathcal{S}_2 \setminus T) > v_2(T \mid \mathcal{S}_2 \setminus T)$ while $p(S) \leq p(T)$. Then player 2 prefers to swap T for S and has enough budget to do so, in contradiction to the fact that he gets his demanded set in the CE.

For the other direction, consider a pair (\mathcal{S}, p) such that Condition (1) holds for every S, T as in the proposition statement. Assume by way of contradiction that (wlog) player 2 is not allocated his demanded set. Then since player 2's budget is exhausted, this means there must be bundles $S \subseteq \mathcal{S}_1$ and $T \subseteq \mathcal{S}_2$ such that $v_2(S \mid \mathcal{S}_2 \setminus T) > v_2(T \mid \mathcal{S}_2 \setminus T)$ and $p(S) \leq p(T)$. Therefore, $v_1(S \mid \mathcal{S}_1 \setminus S) \leq v_1(T \mid \mathcal{S}_1 \setminus S)$, and so by swapping S, T in the allocation we arrive at a new allocation strictly preferred player 2, and no worse for player 1, in contradiction to the Pareto optimality of \mathcal{S} . \square

4.2 Sufficient condition for CE existence and the second welfare theorem.

The second fundamental theorem of welfare economics states [42, Part III, p. 308]:

“[A]ny Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.”

In particular, this means that any socially-efficient allocation that a social planner deems desirable for its equitability can be realized in equilibrium. In our context, such a theorem would say that for every set of players and their preferences, for every PO allocation \mathcal{S} of items among them, we can find budgets for the players and prices for the items which support \mathcal{S} as a CE. Such a theorem does not hold for general preferences even with 2 players (Corollary 3.7). In this section we establish the second welfare theorem for two players with additive preferences.

In our theorem, we shall use prices derived from the additive preferences by a weighted linear combination:

Definition 4.2. Consider 2 additive preferences v_1, v_2 , and parameters $\alpha, \beta \in \mathfrak{R}_+$ such that $\max\{\alpha, \beta\} > 0$. The *combination pricing* p with parameters α, β is an item pricing that assigns every item j the price $p_j = \alpha v_1(\{j\}) + \beta v_2(\{j\})$.

Observe that by additivity of v_1, v_2 in Definition 4.2, the combination pricing p with parameters α, β assigns every bundle S the price $p(S) = \alpha v_1(S) + \beta v_2(S)$. Note that identical items have identical prices (items j, j' are identical precisely if $v_i(\{j\}) = v_i(\{j'\})$ for every player i).

LEMMA 4.3 (BUDGET-EXHAUSTING COMBINATION PRICING IS SUFFICIENT). *Consider 2 players with additive preferences and budgets $b_1 \geq b_2 > 0$ (possibly equal). If for a PO allocation \mathcal{S} there exists a budget-exhausting combination pricing p , then (\mathcal{S}, p) is a CE.*

PROOF. The existence of a budget-exhausting pricing indicates that both players are allocated nonempty bundles in \mathcal{S} . Thus by Proposition 4.1, to prove the lemma it is sufficient to show that Condition (1) holds. For additive preferences this condition can be written as: for every $i, k \in \{1, 2\}, i \neq k$, and for every two bundles $S \subseteq \mathcal{S}_i, T \subseteq \mathcal{S}_k$,

$$v_i(S) > v_i(T) \text{ and } v_k(S) > v_k(T) \implies p(S) > p(T).$$

Plugging in the combination pricing, for every S, T such that $v_1(S) > v_1(T)$ and $v_2(S) > v_2(T)$, it holds that $p(S) = \alpha v_1(S) + \beta v_2(S) > \alpha v_1(T) + \beta v_2(T) = p(T)$. So Condition (1) holds for any $i \neq k \in \{1, 2\}$, and (S, p) is a CE. \square

COROLLARY 4.4 (SECOND WELFARE THEOREM). *Consider 2 players with additive preferences. For every PO allocation S , there exist budgets b_1, b_2 and prices p for which (S, p) is a CE.*

Moreover, if both players are allocated non-empty bundles in S , then (S, p) is a CE for any combination pricing p and corresponding budgets $b_1 = p(S_1), b_2 = p(S_2)$.

PROOF. If S allocates all items to a single player, wlog player 1, then we can get a CE by pricing every item at some arbitrary price $\rho > 0$, setting $b_1 = m\rho$, and setting $b_2 < \rho$ (the budgets can of course be normalized). Otherwise, fix any combination pricing p ; in particular, a combination pricing with parameters $\alpha = \beta = 1$. Set $b_i = p(S_i)$ for every player i . By Lemma 4.3, (S, p) is a CE, completing the proof. \square

4.3 CE existence results for generic budgets.

In this section we ask a different question: given a market with fixed (generic) budgets, is a CE guaranteed to exist?

4.3.1 Overview. Existence of an equilibrium is a fundamental question in the study of markets. The main challenge in answering this question is identifying a PO allocation which, given the budgets, can be supported by appropriate prices in a CE.

One natural candidate we rule out is the PO allocation with maximum Nash social welfare (see Claim B.1). Another candidate is the budget-proportional allocation (defined in Section 2.2), *if one exists* – we pursue this direction in Section 4.3.2. Otherwise, in Section 4.3.3 we define two PO allocations that are “as close as possible” to budget-proportionality, illustrating their properties via Figures 1-2. In Section 4.3.4 we prove our main technical result, which shows sufficient conditions for one of these two PO allocations to be supported by a CE (Theorem 4.10). We illustrate the main technical proof via Figures 3-5. In Sections 4.3.5 and 4.3.6 we show two natural cases in which the sufficient conditions of our technical result hold, guaranteeing the existence of a CE.

All in all, we establish the existence of a CE with generic budgets in three different cases of interest – when a budget-proportional allocation exists, when players are symmetric, and when players’ budgets are almost equal.

4.3.2 Existence Case 1: Budget-proportional allocation. We begin by observing that when valuations are normalized (as we assume in this section), player i gets his budget-proportional share precisely when $v_i(S_i) \geq b_i$. We say that player i gets *at most* his budget-proportional share if $v_i(S_i) \leq b_i$, and we call an allocation *anti-proportional* if every player gets at most his budget-proportional share (Claim B.2 in the appendix demonstrates the existence of markets in which the allocation of every CE is anti-proportional).

We now show that if a budget-proportional allocation exists then a CE exists: observe that every budget-proportional allocation is dominated by a budget-proportional PO allocation, and we show that every such PO allocation is supported in a CE. By the same argument, anti-proportional PO allocations are also supported in a CE.

PROPOSITION 4.5. *For 2 players with additive preferences and arbitrary (possibly equal) budgets, every budget-proportional PO allocation is supported in a CE. Additionally, any anti-proportional PO allocation is supported in a CE.*

The proof is by exploiting (anti-)proportionality to construct a budget-exhausting combination pricing, and it appears in Appendix C.

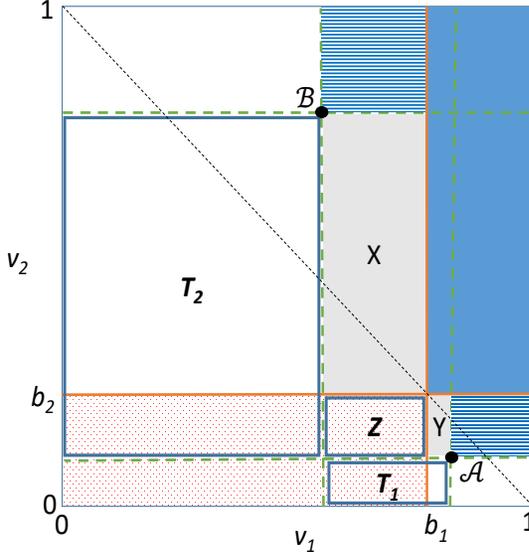


Fig. 1. This figure presents the main setting we analyze in Section 4.3 and some notation we use. It shows the value of an allocation for player 1 on the v_1 -axis and the value of an allocation for player 2 on the v_2 -axis. Every allocation S can be represented by the point $(v_1(S_1), v_2(S_2))$ on the $(v_1 \times v_2)$ -plane. The players' budgets b_1, b_2 are shown on the same axes. The closure of the solid blue area (at or above b_2 and at or to the right of b_1) includes all allocations that are budget-proportional, and is empty by assumption. The closure of the red dotted area represents anti-proportional allocations and has no PO allocations by assumption. Let \mathcal{A}, \mathcal{B} be two PO allocations. By Pareto optimality, the blue striped areas to the right and above \mathcal{A} and \mathcal{B} are both empty (the only allocation in their closures are \mathcal{A} and \mathcal{B}). The figure also marks rectangles T_1, T_2, X, Y, Z which will play a role in our arguments.

COROLLARY 4.6. *If there exists a budget-proportional allocation then a CE exists.*

4.3.3 *As close as possible to budget-proportionality.* For the remainder of the section we assume that neither a budget-proportional nor an anti-proportional PO allocation exist. Figure 1 depicts the setting and fixes our notation.

We introduce two fairness-related definitions, which will help formulate sufficient conditions for existence below. When it is not possible to give each player his budget-proportional share, the next best thing is to give a player the best he can obtain in any PO allocation in which he is allocated at most his budget-proportional share. Due to the indivisible nature of the items, this is *not* necessarily his budget-proportional share; we name it *truncated* share. Formally, denoting the set of PO allocations for preferences v_1, v_2 by $\widetilde{\text{PO}} = \widetilde{\text{PO}}(v_1, v_2)$:

Definition 4.7 (Truncated share). Let $b_i^- = \max_{S \in \widetilde{\text{PO}} | v_i(S_i) \leq b_i} \{v_i(S_i)\}$ be the maximum share player i can obtain in any PO allocation in which he gets at most his budget-proportional share. Denote by $\hat{S}^i = \hat{S}^i(b_i)$ the maximizing PO allocation, i.e., $b_i^- = v_i(\hat{S}^i)$.¹² An allocation S gives player i his truncated budget-proportional share, or *truncated share* for short, if $v_i(S_i) \geq b_i^-$.

¹²The allocation $\hat{S}^i(b_i)$ is well-defined as it is possible to give nothing to player i , so the maximum is taken over a non-empty set of allocations.

Analogously, we define a player's augmented share, his minimum share that is at least his share in any PO allocation:

Definition 4.8 (Augmented share). Let $b_i^+ = \min_{\mathcal{S} \in \widetilde{\text{PO}} | v_i(\mathcal{S}_i) \geq b_i} \{v_i(\mathcal{S}_i)\}$ be the minimum share player i can obtain in any PO allocation in which he gets at least his budget-proportional share. Denote by $\check{\mathcal{S}}^i = \check{\mathcal{S}}^i(b_i)$ the minimizing PO allocation, i.e., $b_i^+ = v_i(\check{\mathcal{S}}_i^i)$. An allocation \mathcal{S} gives player i his *augmented share* (and thus in particular his truncated share) if $v_i(\mathcal{S}_i) \geq b_i^+$.

The following lemma shows a simple but important fact about allocations $\hat{\mathcal{S}}^i, \hat{\mathcal{S}}^k, \check{\mathcal{S}}^i, \check{\mathcal{S}}^k$ (the allocations that give players i, k their truncated or augmented shares, respectively). Namely, for the two players i and k it turns out that necessarily $\hat{\mathcal{S}}^i = \check{\mathcal{S}}^k$ and $\hat{\mathcal{S}}^k = \check{\mathcal{S}}^i$. Thus, this pair of allocations both give each player at least his truncated share (if not his augmented one).

LEMMA 4.9. *Consider 2 players $i \neq k \in \{1, 2\}$ with additive preferences and arbitrary budgets. Assume there are no budget-proportional allocations nor PO anti-proportional allocations. Then the PO allocation $\hat{\mathcal{S}}^i$ coincides with the PO allocation $\hat{\mathcal{S}}^k$. That is, $\hat{\mathcal{S}}^i$ obtains share b_i^+ for player i and share b_k^- for player k .*

PROOF SKETCH. See Figure 2. □

4.3.4 Main technical result. Theorem 4.10 presents two conditions that together are sufficient for a CE to exist: Let i be one of the players, and let v_1, v_2 be the additive valuations. The first condition is genericity of the budgets, defined as *not* belonging to some finite set of budget pairs $R_i(v_1, v_2)$. While somewhat technical, the definition of $R_i(v_1, v_2)$ becomes clearer when observing its role in the proof of Theorem 4.10. The second condition is that some “rectangle of allocations” $T_i = T_i(b_i, v_1, v_2)$, depicted in Figure 1, is empty. We now give formal definitions:

- Let $d = |\widetilde{\text{PO}}|$. Order all allocations in $\widetilde{\text{PO}}$ by player i 's preference, such that his r -th least preferred PO allocation is at index $r \leq d$. Denote this allocation by $\mathcal{S}(r)$, so that $\mathcal{S}(r+1)_i \succ_i \mathcal{S}(r)_i$ for every index $r \leq d-1$. The budget pair in which player i 's budget is b_i (and the other player's budget is $b_k = 1 - b_i$) belongs to $R_i(v_1, v_2)$ if there exists an index r such that $\frac{b_i}{v_i(\mathcal{S}(r+1)_i)} = \frac{1-b_i}{1-v_i(\mathcal{S}(r)_i)}$. Note that there can be at most $d \leq 2^m$ budget pairs in $R_i(v_1, v_2)$.
- Let $T_i = T_i(b_i, v_1, v_2)$ be the set of allocations \mathcal{S} satisfying $v_i(\hat{\mathcal{S}}_i^i) < v_i(\mathcal{S}_i) < v_i(\hat{\mathcal{S}}_i^k)$ and $0 < v_k(\mathcal{S}_k) < v_k(\hat{\mathcal{S}}_k^k)$ (where $\hat{\mathcal{S}}^i = \hat{\mathcal{S}}^i(b_i), \hat{\mathcal{S}}^k = \hat{\mathcal{S}}^k(b_k)$ are as defined in Definition 4.7). T_1 and T_2 are illustrated in Figure 1 for allocations $\mathcal{A} = \check{\mathcal{S}}^1$ and $\mathcal{B} = \check{\mathcal{S}}^2$.

THEOREM 4.10. *Consider 2 players with additive preferences v_1, v_2 and budgets $b_1 > b_2$. Assume there are no budget-proportional allocations nor PO anti-proportional allocations. If for some player i , $(b_1, b_2) \notin R_i(v_1, v_2)$ and the set $T_i = T_i(b_i, v_1, v_2)$ is empty, then a CE exists. Moreover, in this CE every player gets his truncated share.*

The suffix of Theorem 4.10 is interesting in light of Example C.4 in Appendix C, which shows that not every CE gives every player his truncated share. The proof of Theorem 4.10 appears in Appendix C.3.

Necessity of the condition that T_i is empty. Our simulations identified an example in which for each player i , T_i is not empty, and there is no CE with item prices based on scaling $v_i(\{j\})$ (other CEs were found).¹³ This implies that dropping this condition would require new techniques.

¹³The example includes 7 items. Player 1's values are (0.1420, 0.0808, 0.1921, 0.1717, 0.1651, 0.1200, 0.1283), player 2's values are (0.0827, 0.1056, 0.1743, 0.1515, 0.1862, 0.1123, 0.1874). The budgets are 0.8093 and 0.1907.

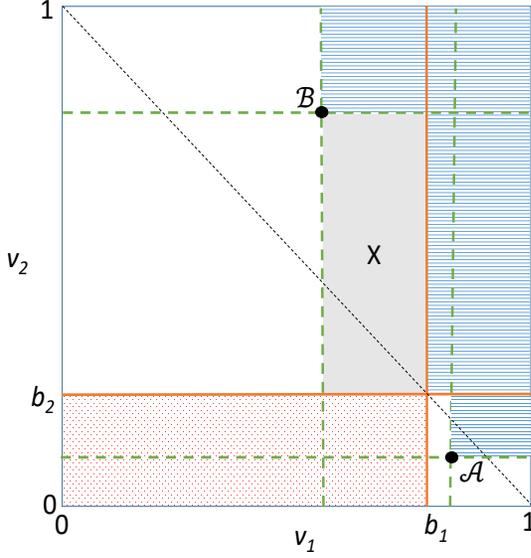


Fig. 2. This figure presents the proof of Lemma 4.9, using the notation of Figure 1, for the case of $k = 1, i = 2$ (the case of $k = 2, i = 1$ is similar). Let $\mathcal{A} = \hat{S}^1$ and $\mathcal{B} = \hat{S}^2$ (both PO allocations). Observe that indeed in allocation \mathcal{A} player 1 is receiving value above b_1 and in allocation \mathcal{B} player 2 is receiving value above b_2 . In Figure 1 we have seen that the closure of the blue striped area does not include any allocation but \mathcal{A} and \mathcal{B} . We now argue that $\mathcal{B} = \hat{S}^2 = \hat{S}^1$ (showing $\mathcal{A} = \hat{S}^1 = \hat{S}^2$ is similar). By definition of $\mathcal{B} = \hat{S}^2$, it is the lowest PO allocation at or above b_2 . Any PO allocation at or to the left of b_1 that is to the right of \mathcal{B} must be in the interior of the gray rectangle X , as there are no PO anti-proportional allocations (the closure of the dotted red area is empty from PO allocations). Yet such a point in the interior of X is not only to the right of \mathcal{B} , it is also below it. This means that it is closer than \mathcal{B} to b_2 from above, yielding a contradiction. The closed rectangle X must therefore be empty of PO allocations except for \mathcal{B} (and thus must also be empty of non-PO allocations). But the point corresponding to allocation \hat{S}^1 must fall within closure of rectangle X by definition (it is the rightmost PO point at or to the left of b_1 , and it cannot lie to the left of \mathcal{B}). Thus $\hat{S}^1 = \mathcal{B}$, completing the proof.

4.3.5 *Existence Case II: Symmetric players.* A direct corollary of Theorem 4.10 is that if every allocation is PO and budgets are generic, then a CE exists:

PROPOSITION 4.11. *Consider 2 players with additive preferences and budgets $b_1 > b_2$ such that $(b_1, b_2) \notin R(v_1, v_2)$. If every allocation is PO then there exists a CE.*

In the symmetric case where both players share the same additive preference, we have a “constant-sum game” and thus every allocation is PO. In addition, an anti-proportional allocation in which some player gets less than his truncated share cannot exist. The next corollary follows:

COROLLARY 4.12. *Consider 2 players with additive preferences and budgets $b_1 > b_2$ such that $(b_1, b_2) \notin R(v_1, v_2)$. If both players share the same preference (i.e., are symmetric), then there exists a CE that gives every player his truncated share.*

4.3.6 *Existence Case III: Almost equal budgets.* To fairly split indivisible items between two a priori equal players, CEs with equal budgets are natural candidates, but do not always exist. We show that breaking the symmetry by using unequal but very close budgets is sufficient to

ensure existence of a CE for additive preferences and any number of items. The proof is deferred to Appendix C.

PROPOSITION 4.13. *Consider 2 players with additive preferences and budgets $b_1 > b_2$. For sufficiently small $\epsilon > 0$, if $b_1 - b_2 \leq \epsilon$ then there exists a CE that gives every player his truncated share.*

5 SUMMARY AND OPEN QUESTIONS.

Our main conceptual contribution in this paper is to show that to get positive equilibrium existence results for interesting classes of markets, it is sufficient to exclude degenerate market instances. This is done by adding small noise to the players' budgets to make them generic (in the spirit of smoothed analysis for getting positive computational tractability results [55]). Unlike Budish's approach [18], there is no need to relax the equilibrium notion, and our model allows for possibly very different budgets. The classes of markets for which this is shown to work are: (1) markets with general ordinal preferences and a small number of items; (2) markets with 2 cardinal additive players who have almost equal budgets, or symmetric preferences, or for which a proportional allocation exists; (3) markets with 2 ordinal leveled players or n ordinal lexicographic players. We expect additional classes of markets to be added to this list in the future.

We make two other main contributions: We define some of the first fairness notions applicable to the allocation of indivisible items among players with different entitlements, and show the connection between these notions and market equilibria. Additionally, we initiate the study of a hierarchy of strict preference classes, where interestingly the highest level is representable by submodular valuations. (A similar study of a hierarchy of valuation classes has been key in the combinatorial auctions literature). Many exciting open directions remain, including:

- Is CE existence guaranteed with generic budgets for all 2-player, additive markets? We view our second welfare theorem in Corollary 4.4 as a positive indication. The same question applies to more general responsive preferences, for which our CE characterization in Proposition 4.1 holds.
- Is CE existence guaranteed with generic budgets for all markets with symmetric preferences?
- Why does CE existence seem common in practice? Does it hold with high probability for some natural distribution over preferences?
- What is the "right" notion of envy-freeness (as opposed to fair share notions) for budgeted players and indivisible items?
- For which classes of markets are fair allocations according to our notions guaranteed to exist with generic budgets?

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A CLASSES OF ORDINAL PREFERENCES.

A CE does not always exist for general ordinal preferences even when budgets are generic, so in this section we focus on more restricted classes of ordinal preferences. In Section A.1 we initiate the exploration of a hierarchy of monotone and strict ordinal preference classes (a similar exploration of valuation classes in the quasi-linear model has been central in facilitating positive results – see, e.g., [38]). Interestingly, the highest class in the hierarchy turns out to be representable by *submodular* valuations. In Section A.2 we present a class of preferences that we call “leveled”, for which a CE is guaranteed to exist for 2 players with generic budgets.

A.1 A hierarchy of ordinal preferences.

In this section we establish the following hierarchy:

$$\text{LEXICOGRAPHIC} \subsetneq \text{ADDITIVE} \subsetneq \text{RESPONSIVE} \subsetneq \text{SUBMODULAR} = \text{GENERAL}. \quad (2)$$

We also show that existence of a CE is guaranteed for the lowest level in the hierarchy.

Definitions. GENERAL is the class of all monotone and strict ordinal preferences. ADDITIVE (resp., SUBMODULAR) is the subclass of preferences in GENERAL that can be represented by an additive (resp., a submodular) valuation.¹⁴ LEXICOGRAPHIC (resp., RESPONSIVE) is the subclass of preferences in GENERAL that are lexicographic (resp., responsive), according to the following definitions:

Definition A.1. A monotone and strict ordinal preference $>$ is *lexicographic* if there exists a strict preference $>^*$ over items such that $S > T \iff \max_{>^*} \{s \in S \setminus T\} >^* \max_{>^*} \{t \in T \setminus S\}$ (i.e., the most preferred item in $S \setminus T$ is preferred over the most preferred item in $T \setminus S$).

Definition A.2. A monotone and strict ordinal preference $>$ is *responsive* if there exists a strict preference $>^*$ over items such that $\forall S, \forall j, j' \notin S, S \cup \{j\} > S \cup \{j'\} \iff j >^* j'$.

Intuitively, a player with a lexicographic preference would trade all of his items for a single higher-ranked item; a player with a responsive preference would always trade a lower-ranked item for a higher-ranked one. Responsive preferences arise naturally in the matching literature (see, e.g., [52]). For example, consider a hospital that ranks individual doctors based on their test scores, then the hospital’s preference over sets of doctors will be responsive to its preference over individuals.

Establishing the hierarchy. Propositions A.3 and A.4 establish the hierarchy in Equation (2).

PROPOSITION A.3. *Every monotone and strict ordinal preference $<$ can be represented by a submodular valuation; that is, SUBMODULAR = GENERAL.*

PROOF. Index the bundles such that $\emptyset < S_1 < S_2 < \dots < S_{2^m-1}$. Define

$$v(S_t) = 1 - 2^{-t}.$$

¹⁴A valuation v is *submodular* if for every two bundles S, T , $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$. We do not consider the class of preferences represented by *unit-demand* valuations (which assign a value to each item, and value every bundle by the highest-valued item within it), since such valuations are inherently non-strict.

It is not hard to see that v represents $<$. We now argue that v is submodular. Consider the marginal value of an item $j \notin S_t$ given a bundle S_t . Using monotonicity we have that

$$v(j | S_t) = v(S_t \cup \{j\}) - v(S_t) \geq v(S_{t+1}) - v(S_t) = 2^{-t} - 2^{-(t+1)} = 2^{-(t+1)}. \quad (3)$$

Since all values are strictly less than 1,

$$v(j | S_t) = v(S_t \cup \{j\}) - v(S_t) < 1 - (1 - 2^{-t}) = 2^{-t}. \quad (4)$$

Let $S_{t'}$ be a superset of S_t that does not include item j ; due to monotonicity, $t' \geq t + 1$. By Inequality (4), $v(j | S_{t'}) < 2^{-t'} \leq 2^{-(t+1)}$, and so by Inequality (3), $v(j | S_{t'}) < v(j | S_t)$. We have established the property of decreasing marginals, so v is submodular as claimed. \square

PROPOSITION A.4. *The strict inclusion relations in Equation (2) hold.*

PROOF.

- **LEXICOGRAPHIC \subsetneq ADDITIVE.** Let $<$ be any preference in LEXICOGRAPHIC induced by a preference $<^*$ over items. Wlog assume $1 <^* 2 <^* \dots <^* m$. Define an additive valuation v in which item j has value 2^{j-1} (larger than the total value of all items ranked lower than j). Observe that v represents the preference $<$. Thus LEXICOGRAPHIC \subseteq ADDITIVE, and it is not hard to see there are preferences in ADDITIVE \setminus LEXICOGRAPHIC.
- **ADDITIVE \subsetneq RESPONSIVE.** It is not hard to see that ADDITIVE \subseteq RESPONSIVE; we now construct a preference in RESPONSIVE \setminus ADDITIVE. Let $>$ be a preference in RESPONSIVE induced by the preference $A >^* B >^* C >^* D >^* E$ over items. The preference $>^*$ over items induces a partial order over bundles of the same cardinality. For bundles of size 2, we know that either $AB > AC > BC > AD > BD > CD > AE > BE > CE > DE$, or $AB > AC > BC > AD > BD > AE > CD > BE > CE > DE$. However, $>^*$ does not induce an order over bundles of different cardinalities, which may be arbitrary subject to monotonicity. We may thus assume that both $AD > BCD$ and $BCE > AE$, without violating the responsiveness or monotonicity of preference $>$. Now assume for contradiction that preference $>$ is represented by an additive valuation v . Then $AD > BCD \implies v(A) > v(BC)$, but $BCE > AE \implies v(BC) > v(A)$, contradiction.
- **RESPONSIVE \subsetneq GENERAL.** It is not hard to see there are preferences in GENERAL \setminus RESPONSIVE. \square

CLAIM A.5. *For n players with lexicographic preferences and distinct budgets, a CE exists.*

PROOF. Run the well-known *serial dictatorship* allocation protocol (e.g., [1]), as follows: Order the players by decreasing budget and let each player $i < n$ in turn pick his most preferred item among the unallocated items. Price this item at b_i (player i 's budget). Allocate all leftover items to player n whose budget is smallest, pricing such items at b_n/r where r is the number of leftovers. It is not hard to see that the resulting allocation and item pricing is a CE.¹⁵ \square

A.2 Existence of CE for 2 players with leveled preferences.

LEVELED is the subclass of preferences in GENERAL that are leveled:

Definition A.6. A monotone and strict ordinal preference $>$ is *leveled* if for every two bundles S, T such that $|S| > |T|$, $S > T$.

¹⁵The same argument establishes CE existence for unit-demand preferences, although they are not strict.

Table 1. Fair share and envy-free notions for ordinal preferences (our contribution in bold).

	Equal budgets	Arbitrary budgets
Divisible items	Fair share (FS) [56] Envy-freeness (EF) [31]	Budget-FS Budget-EF ^a
Indivisible items	1-out-of- n maximin share [18] EF-1 [18], EF-1* [20]	ℓ-out-of-d maximin share (Definition 2.5) Justified-EF (Definition B.7), open direction

^aBudget-FS and budget-EF are natural generalizations of FS and EF, defined in Appendix B.1.

Table 2. Fair share notions for cardinal preferences (our contribution in bold).

	Equal budgets	Arbitrary budgets
Divisible items	Proportionality [56]	Budget-proportionality [13, 51]
Indivisible items	Proportionality [56]	Truncated share (Definition 4.7)

A player with a leveled preference prefers a larger bundle over any smaller bundle. Among bundles of the same cardinality there can be any preference ordering. LEVELED is incomparable to LEXICOGRAPHIC, ADDITIVE and RESPONSIVE, in the sense that none of these classes contains the other.¹⁶ In our model, unlike the quasi-linear one, leveled preferences are sufficient to ensure CE existence for 2 players:¹⁷

PROPOSITION A.7. *For 2 players with leveled preferences and budgets $b_1 > b_2$, a CE exists provided that $m \cdot b_1$ is not an integer (where m is the number of items).*

PROOF. Let $p^* = 1/m$, $k_1 = \lfloor mb_1 \rfloor$ and $k_2 = \lfloor mb_2 \rfloor$. Intuitively, p^* is the “fair” price for an item, and k_1, k_2 are the target number of items to allocate to each player in order to achieve proportionality. By assumption, $m \cdot b_1$ is not an integer and so $k_1 + k_2 = m - 1$. By definition, $b_1/k_1 > p^* > b_1/(k_1 + 1)$, and $b_2/k_2 > p^* > b_2/(k_2 + 1)$.

Assume wlog that $b_1/(k_1 + 1) > b_2/(k_2 + 1)$. Give player 2 his most preferred set of k_2 items, and player 1 the remaining $k_1 + 1$ items. Price each of player 1’s items at $b_1/(k_1 + 1)$, and each of player 2’s items at b_2/k_2 .

We argue that no player wishes to deviate: Player 1 would only want to deviate to a bundle with $\geq k_1 + 1$ items, but all such bundles are above his budget – he is currently exhausting his budget with items priced $b_1/(k_1 + 1)$, and all other items cost $b_2/k_2 > p^* > b_1/(k_1 + 1)$. Player 2 gets his most preferred set of k_2 items, so would only want to deviate to a bundle with $> k_2$ items, but all such bundles are above his budget – item prices are at least $b_1/(k_1 + 1) > b_2/(k_2 + 1)$. \square

B CE AND FAIRNESS.

B.1 Detailed fairness preliminaries.

The discussion in this section is summarized in Tables 1 and 2, which also show where our fairness concepts fit in with some of the existing concepts.

¹⁶The construction in Proposition A.4 which separates RESPONSIVE and ADDITIVE is a responsive preference that is not leveled. To see the other direction, observe that the following leveled preference over 3 items is not responsive: $ABC > BC > AB > AC > C > B > A > \emptyset$. This is because $AB > AC$, but $C > B$.

¹⁷Consider 2 quasi-linear players and 2 items A, B . Player 1 has values $(2, 2, 2)$ for bundles (A, B, AB) , and player 2 has values $(0, 0, 3)$ for these bundles. These valuations represent leveled preferences, but one can verify that in the quasi-linear model with value for money, a CE does not exist in this setting.

B.1.1 Ordinal preferences. Fair share and envy-freeness are well-defined for ordinal preferences when items are *divisible*. Intuitively, fair share guarantees that each player believes he receives at least $1/n$ of the “cake” being divided, and envy-freeness guarantees he believes no one else receives a better slice than him. More formally, *fair share* (FS) requires for each player to receive a bundle that he prefers at least as much as the bundle consisting of a $1/n$ -fraction of every item on the market. Player i *envies* player k given allocation \mathcal{S} if $S_i <_i S_k$, and an allocation is *envy-free* (EF) if no player envies another player.

We observe that for divisible items, both notions extend naturally to budgeted players: *Budget-FS* requires every player to prefer his bundle at least as much as a b_i -fraction of the bundle of all items. Player i envies player k with a larger budget if he prefers a b_i/b_k -fraction of S_k to S_i , and player k' with a smaller budget if he prefers $S_{k'}$ to a $b_{k'}/b_i$ -fraction of S_i ; the *budget-EF* property excludes such envy.

When items are indivisible, FS is not well-defined; EF is well-defined but often cannot be satisfied [27]. To circumvent the definition and existence issues stemming from indivisibilities, Budish [18] proposes appropriate variants: 1-out-of- n maximin share (Definition 2.5) and envy-free up to one good (Definition 2.6). Caragiannis et al. [20] introduce a strengthening of EF-1 called EF-1* (or EFX), envy-freeness up to *any* good, in which for every players i, k and *any* item $j \in S_k$ it holds that $S_k \setminus \{j\} <_i S_i$.

We discuss how to generalize the maximin share guarantee for budgeted players in Section 3.1, defining the notion of ℓ -out-of- d maximin share (Definition 2.5). We also define a notion called justified-EF for budgets (Definition B.7), and leave the question of how to generalize (non-justified) EF to budgets as an open direction.

Application to CEs. It is well-known that every CE with n equal budgets gives every player his 1-out-of- n maximin share and achieves EF. Budish [18] shows that every CE with almost equal budgets guarantees 1-out-of- $(n + 1)$ maximin share (whereas 1-out-of- n maximin share cannot always be guaranteed). Proposition 3.2 generalizes this to arbitrary budgets using the notion of ℓ -out-of- d maximin share.

Budish [18] also shows that every CE with almost equal budgets is EF-1 (a short proof appears for completeness in Proposition B.3). We demonstrate that a CE with almost equal budgets is not necessarily EF-1* (Claim B.6), but any CE with arbitrary budgets is justified-EF (Claim B.8).

B.1.2 Cardinal preferences. For cardinal preferences, the parallel of FS is the notion of proportionality, which extends naturally to players with different budgets: Given a budget profile b , an allocation \mathcal{S} gives player i his *budget-proportional share* if player i receives at least a b_i -fraction of his value for all items, that is $v_i(\mathcal{S}_i) \geq b_i \cdot v_i(M)$. An allocation is *budget-proportional* (a.k.a. weighted-proportional) if every player receives his proportional share. When all budgets are equal, such an allocation is simply called *proportional*. Budget-proportional allocations play a central role in our positive results in Section 4. It is clear that a budget-proportional allocation does not always exist with indivisible items (e.g., with a single one). In Section 4.3.3 we present a relaxation of budget-proportional share that we call truncated share, which is guaranteed to exist (Definition 4.7).

Budget-EF also extends naturally to players with cardinal preferences and budgets, by saying that player i envies player k if $v(S_i) < b_i \cdot v(S_k)/b_k$. It is not hard to see that budget-EF implies budget-proportionality. Another fairness notion for cardinal preferences which naturally extends to budgets (e.g., [16]) is *Nash social welfare* maximization. Given a budget profile b , an allocation \mathcal{S} is Nash social welfare maximizing if it maximizes $\prod_i (v_i(\mathcal{S}_i))^{b_i}$, or equivalently, $\sum_i b_i \log v_i(\mathcal{S}_i)$, among all allocations (notice that the maximizer is invariant to budget or valuation scaling).

Application to CEs. In all settings for which we prove CE existence in Section 4.3, the CEs guarantee every player his truncated share. A CE with equal budgets maximizes the (unweighted) Nash social welfare; in contrast, we show a simple market with different budgets in which a CE exists, but the allocation that maximizes Nash social welfare is not supported by a CE:

CLAIM B.1. *There exists a market of 2 items and 2 players with additive preferences and unequal budgets, such that the unique PO allocation that maximizes (weighted) Nash social welfare is not supported in a CE, but a CE exists.*

PROOF. Consider 2 players Alice and Bob with budgets 101, 100, and 2 items A, B valued by Alice $v_1(A) = 5, v_1(B) = 4$ and by Bob $v_2(A) = 1000, v_2(B) = 1$. Clearly the only CE gives item A to Alice and item B to Bob (since Bob cannot prevent Alice from taking item A even if he pays his full budget for it), but the only allocation maximizing the Nash social welfare (i.e., maximizing $101 \log v_1(S_1) + 100 \log v_2(S_2)$) gives item A to Bob and item B to Alice. \square

CLAIM B.2. *There exists a market of 2 items and 2 players with unequal budgets, for which a CE exists and every CE allocation is anti-proportional.*

PROOF. Let $b_1 = 5/8, b_2 = 3/8, v_1(A) = 100, v_1(B) = 101, v_2(A) = 1, v_2(B) = 1000$. Since $b_1 > b_2$, in every CE player 1 gets his preferred item, item B . Moreover, he cannot get both items, as $b_2 > b_1/2$ so player 2 can always afford at least one item. So in every CE, player 1 gets item B and player 2 gets item A , and their shares are $100/201 < 5/8$ and $1/1001 < 3/8$, respectively. Equilibrium prices that support this allocation are $p(A) = 3/8, p(B) = 5/8$. \square

B.2 Variants of envy-freeness.

Budish [18] establishes an envy-free property of CEs with almost equal budgets, as follows (we include a short proof for completeness):

PROPOSITION B.3 (BUDISH [18]). *Consider a CE (S, p) with almost equal budgets $b_1 \geq b_2 \geq \dots \geq b_n \geq \frac{m-1}{m}b_1$, then the CE allocation S is EF-1.*

PROOF. Fix any two players i, k . We show that there is some item $j^* \in S_k$ such that $S_i \succ_i S_k \setminus \{j^*\}$. By Claim 2.4, we may assume without loss of generality that the budget of player k is exhausted. So there exists some item $j^* \in S_k$ such that its price p_{j^*} is at least $b_k/|S_k| \geq b_k/m$. The price of $S_k \setminus \{j^*\}$ is therefore at most $\frac{m-1}{m}b_k \leq b_n \leq b_i$, and so i can afford $S_k \setminus \{j^*\}$. Since S is a CE allocation and bundle $S_k \setminus \{j^*\}$ is within i 's budget, $S_i \succ_i S_k \setminus \{j^*\}$ as needed. \square

A requirement stronger than (implying) EF-1 and weaker than (implied by) EF is the following:

Definition B.4 (Caragiannis et al. [20], Definition 4.4). An allocation S is EF-1^{*} if for every two players i and k , for every item $j \in S_k$ it holds that $S_k \setminus \{j\} \prec_i S_i$.

An EF-1^{*} allocation always exists for 2 players – the cut-and-choose procedure from cake-cutting results in such an allocation. It is an open question whether it always exists in general. We demonstrate (by an example with non-strict preferences) that an EF-1^{*} allocation is not necessarily EF even when an EF allocation exists in the market:

Example B.5. Consider 3 symmetric additive players, 22 “small” items worth 1 each, and 2 “large” items worth 7 each. An EF allocation is two bundles of 1 large item and 5 small items each, and one bundle of 12 small items. An EF-1^{*} allocation that is not EF is one bundle of 2 large items, and two bundles of 11 small items each.

While Proposition B.3 shows that a CE implies EF-1 for almost equal budgets, we next prove that a CE does not imply the stronger property EF-1^{*}.

CLAIM B.6. *The allocation of a CE from almost equal budgets is not necessarily EF-1*, even for 2 symmetric players with an additive preference over 4 items.*

PROOF. The proof follows from Example C.4. Recall that the allocation $(\{A, B\}, \{C, D\})$ is a CE allocation, but it is not EF-1*: player 2 envies player 1 even if he gives up item B. \square

We conclude this section by defining a notion of envy-freeness that a CE with different budgets guarantees. Borrowing from the matching literature, we define *justified envy* as the envy of a player with a higher budget towards a player with a lower budget. The intuition is that any envy of a lower-budget player towards the allocation of a higher-budget player isn't justified and so "doesn't count", because the higher-budget player "deserves" a better allocation. We thus only care about eliminating justified envy.

Definition B.7. An allocation \mathcal{S} is *justified-EF* given budgets $b_1 \geq \dots \geq b_n$ if for every two players $i < k$, player i (with the higher budget) does not envy player k (with the lower budget). An allocation \mathcal{S} is *justified-EF for coalitions* if for every player i and set of players K such that $i \notin K$ and $b_i \geq \sum_{k \in K} b_k$, player i does not envy K , i.e., $\bigcup_{k \in K} \mathcal{S}_k <_i \mathcal{S}_i$.

CLAIM B.8. *Every CE allocation (with possibly very different budgets) is justified-EF for coalitions.*

PROOF. Assume that $b_i \geq \sum_{k \in K} b_k$. Since the total price $\sum_{k \in K} p(\mathcal{S}_k)$ is at most $\sum_{k \in K} b_k$, player i can afford the bundle $\bigcup_{k \in K} \mathcal{S}_k$. Because \mathcal{S} is a CE allocation, it must hold that $\bigcup_{k \in K} \mathcal{S}_k <_i \mathcal{S}_i$. \square

C MISSING PROOFS.

C.1 Missing proofs from Section 2.

PROOF OF THEOREM 2.3 (FOR COMPLETENESS). Assume for contradiction an alternative allocation \mathcal{S}' , such that for every player i for whom $\mathcal{S}_i \neq \mathcal{S}'_i$, it holds that $\mathcal{S}_i <_i \mathcal{S}'_i$. Consider the total payment $\sum_i p(\mathcal{S}'_i)$ for the alternative allocation given the CE prices p . By market clearance, $\sum_i p(\mathcal{S}'_i) = \sum_i p(\mathcal{S}_i)$. Therefore there must exist a player i for whom $\mathcal{S}_i \neq \mathcal{S}'_i$ but $p(\mathcal{S}'_i) \leq p(\mathcal{S}_i)$. This means that \mathcal{S}_i cannot be demanded by player i , leading to a contradiction. \square

C.2 Missing proofs from Section 3.

PROOF OF PROPOSITION 3.3 (SKETCH). Observe that in any CE, a player whose budget is not among the largest three will remain unallocated. We proceed by case analysis:

- If $b_1 > 3b_2$, then player 1 gets all 3 items, each for a price of $b_1/3 > b_2$.
- If $3b_2 \geq b_1 > 2b_2$, then player 1 gets the bundle of size 2 that he most prefers, and pays $b_1/2 > b_2$ for each item, and player 2 gets the remaining item for price of b_2 .
- If $2b_2 \geq b_1 > b_2 + b_3$, then if the pair of items that player 1 prefers most does not contain player 2's most preferred item, give player 1 this pair and charge him $b_1/2$ for each item, and give player 2 the remaining item for price b_2 . Otherwise, give player 2 his second most preferred item for a price of b_2 , give the other 2 items to player 1 and charge him $b_2 + \epsilon > b_2$ for player 2's most preferred item, and $b_1 - b_2 - \epsilon > b_3$ for the other item.
- If $b_1 = b_2 + b_3$, then if there is an item that player 1 prefers over all pairs of items, he gets the item and pays his budget, while each other player – in the order of their budgets – picks his most preferred item and pays his budget. If there is no such item, player 2 gets his most preferred item for price b_2 , and player 1 get the remaining 2 items, each for a price of $b_1/2 < b_2$.
- If $b_2 + b_3 > b_1 > b_2$, then each player in the order of budgets picks his most preferred item out of the remaining items, paying his budget.

It is not hard to verify that each of these price vectors indeed form a CE for the given budgets. \square

PROOF OF LEMMA 3.5. Allocation \mathcal{S} is PO as both Alice and Bob prefer B over A , and both have the same order over items C, D and E . Thus there is no exchange of items that improves the situation for both players simultaneously. Assume for contradiction that \mathcal{S} is supported in a CE. Since both Alice and Bob prefer Alice's bundle $\{X_1, Y_1, Y_2\}$ over Bob's bundle $\{X_2, Y_3\}$, to support \mathcal{S} in a CE it must be the case that $b_1 > b_2$. By Claim 2.4 we may assume both players exhaust their budgets, and so $b_1 = p_{X_1} + p_{Y_1} + p_{Y_2}$ and $b_2 = p_{X_2} + p_{Y_3}$. In such a CE, since Alice prefers replacing X_1 by Y_3 and replacing $\{Y_1, Y_2\}$ by X_2 , it must be that $p_{X_1} < p_{Y_3}$ and $p_{Y_1} + p_{Y_2} < p_{X_2}$. This implies that $b_1 = p_{X_1} + p_{Y_1} + p_{Y_2} < p_{X_2} + p_{Y_3} = b_2$, a contradiction. \square

PROOF OF PROPOSITION 3.6. Consider Example 3.4. By Theorem 2.3, to show that no CE exists we need only consider PO allocations, in which every item is allocated. By Claim 2.4 we can further assume that if there were a CE, then both players would exhaust their budgets. We begin by observing that since Alice has more money, it is not possible that Bob gets both A and B – Alice can afford Bob's bundle, and she prefers $\{A, B\}$ over $\{C, D, E\}$. On the other hand, Alice cannot get $\{A, B\}$ or any superset of it. This is because in such a CE one of the items A or B will have price at most $b_1/2 < b_2$, and Bob prefers that item over any pair of items from his set, and there is such a pair that costs at least $(2/3)b_2 > b_1/2$, thus Bob will deviate. We are left to consider the case that Alice gets one of A and B , and Bob gets the other. If Alice also gets all three items in $\{C, D, E\}$, the allocation is not PO as it is dominated by giving Alice the set $\{A, B\}$ and Bob the rest. If Alice gets at most one of the items in $\{C, D, E\}$, she would rather exchange bundles with Bob, and can afford it, so it is not a CE. We are left with the case in which Alice gets one of A or B , and two items from the set $\{C, D, E\}$, but Lemma 3.5 rules out the existence of a CE in this case, completing the proof. \square

The next proposition shows existence of CE for 2 players and 4 items. [53] settles the case of more than 2 players, showing existence for 3 players and non-existence for ≥ 4 players.

PROPOSITION C.1 (4 ITEMS). *Consider 4 items and 2 players with monotone strict preferences and budgets $b_1 > b_2$. If $b_1 \notin \{4b_2, 3b_2, 3b_2/2\}$, then a CE exists.*

PROOF. Denote the items by $\{A, B, C, D\}$. We partition the space of budgets by the ratio of b_1 and b_2 as follows:

- If $b_1 > 4b_2$: Price every item at $b_1/4$ and give all items to player 1.
- If $4b_2 > b_1 > 3b_2$: Give player 1 the bundle of size 3 that he most prefers, and price each item at $b_1/3$. Give the leftover item to player 2 at price b_2 .
- If $3b_2 > b_1 > 3b_2/2$: See Claim C.2.
- If $3b_2 > b_1 > 3b_2/2$: See Claim C.3.

CLAIM C.2. *A CE exists if $3b_2 > b_1 > 3b_2/2$.*

PROOF OF CLAIM C.2. If player 1 prefers any triplet of items to any pair, then give player 2 the item that he most prefers at price b_2 , and give player 1 the remaining triplet at price $b_1/3$ per item. Otherwise assume without loss of generality that $\{A, B\}$ is player 1's most preferred pair. If player 1 prefers $\{A, B\}$ only to one triplet, without loss of generality $\{B, C, D\}$, then there are three cases:

- Case 1: Player 2's favorite item is not A . Give player 2 the item that he most prefers at price b_2 , and give player 1 the remaining triplet at price $b_1/3$ per item.
- Case 2: Player 2 prefers $\{C, D\}$ to $\{B\}$. If $b_1 > 2b_2$, give player 1 the pair $\{A, B\}$ and player 2 the pair $\{C, D\}$. Set the prices to be $(b_1 - b_2 + \epsilon, b_2 - \epsilon, b_2/2, b_2/2)$. Otherwise if $b_1 \leq 2b_2$ set the prices to be $(b_2 + \epsilon, b_1 - b_2 - \epsilon, b_2/2, b_2/2)$.

- Case 3: Give player 1 the triplet $\{A, C, D\}$ and player 2 the item B . Set the prices to be $(b_2 + \epsilon, b_2, \frac{b_1 - b_2 - \epsilon}{2}, \frac{b_1 - b_2 - \epsilon}{2})$.

Assume now player 1 prefers $\{A, B\}$ to both $\{A, C, D\}$ and $\{B, C, D\}$. If $b_1 > 2b_2$, then give player 1 the pair $\{A, B\}$ and give player 2 the pair $\{C, D\}$. Set the prices to be $(b_1/2, b_1/2, b_2/2, b_2/2)$. Otherwise $2b_2 \geq b_1$ and there are two cases:

- Case 1: Player 2 prefers $\{C, D\}$ to $\{A\}$ or to $\{B\}$; assume without loss of generality the latter. Then give player 1 the pair $\{A, B\}$ and give player 2 the pair $\{C, D\}$. Set the prices to be $(b_2 + \epsilon, b_1 - b_2 - \epsilon, b_2/2, b_2/2)$.
- Case 2: Player 2 prefers $\{A\}$ and $\{B\}$ to $\{C, D\}$. Assume without loss of generality that player 1 prefers $\{A, C, D\}$ to $\{B, C, D\}$. Give player 1 the triplet $\{A, C, D\}$ and give player 2 the item B . Set the prices to be $(b_2 + \epsilon, b_2, \frac{b_1 - b_2 - \epsilon}{2}, \frac{b_1 - b_2 - \epsilon}{2})$.

□

CLAIM C.3. *A CE exists if $3b_2/2 > b_1 > b_2$.*

PROOF OF CLAIM C.3. Without loss of generality let A be the single item most preferred by player 2. Let $\delta = b_1 - b_2$.

- Case 1: The complement to a pair that player 1 prefers over $\{B, C, D\}$ appears before A in player 2's preference ordering. Give player 1 the most preferred such pair, which must include A by monotonicity, and give player 2 its complement.
 - Subcase (i): Player 1 gets his most preferred pair. Set the prices to be $b_1/2, b_1/2$ for player 1's items, and $b_2/2, b_2/2$ for player 2's items.
 - Subcase (ii): Player 1 gets his second most preferred pair, without loss of generality $\{A, C\}$ where the most preferred pair is $\{A, B\}$. This means that $\{C, D\}$ appears after $\{A\}$ in player 2's preference ordering. Set the prices to be $(b_1 - b_2/2, b_2/2 + \epsilon, b_2/2, b_2/2 - \epsilon)$, where $\epsilon < \delta$.
 - Subcase (iii): Player 1 gets his third most preferred pair, without loss of generality $\{A, D\}$ where the first and second most preferred pairs are $\{A, B\}, \{A, C\}$. This means that $\{C, D\}$ and $\{B, D\}$ appear after $\{A\}$ in player 2's preference ordering. Set the prices to be $(b_1/2 + \delta, b_2/2, b_2/2, b_1/2 - \delta)$.
- Case 2: Player 1 prefers $\{A\}$ over $\{B, C, D\}$. Give player 1 the item A and give player 2 the bundle $\{B, C, D\}$. Set the prices to be $(b_1, b_2/3, b_2/3, b_2/3)$.
- Case 3: Give player 1 the bundle $\{B, C, D\}$ and give player 2 the item A .
 - Subcase (i): Player 1 has one pair preferred over $\{B, C, D\}$, without loss of generality $\{A, B\}$. This means that $\{C, D\}$ appears after $\{A\}$ in player 2's preference ordering. Set the prices to be $(b_2, b_1 - 2\epsilon, \epsilon, \epsilon)$, where $\delta/2 < \epsilon < \delta$.
 - Subcase (ii): Player 1 has two pairs preferred over $\{B, C, D\}$, without loss of generality $\{A, B\}$ and $\{A, C\}$. This means that $\{C, D\}$ and $\{B, D\}$ appear after $\{A\}$ in player 2's preference ordering. Set the prices to be $(b_2, \frac{b_1 - \epsilon}{2}, \frac{b_1 - \epsilon}{2}, \epsilon)$, where $\epsilon < \delta$.
 - Subcase (iii): Player 1 has three pairs preferred over $\{B, C, D\}$, these pairs are $\{A, B\}, \{A, C\}$ and $\{A, D\}$. This means that $\{C, D\}, \{B, D\}$ and $\{B, C\}$ appear after $\{A\}$ in player 2's preference ordering. Set the prices to be $(b_2, b_1/3, b_1/3, b_1/3)$.

□

□

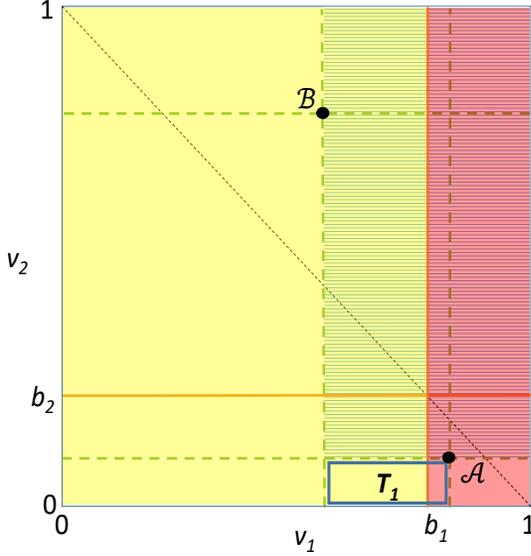


Fig. 3. This figure illustrates the first part of the proof of Theorem 4.10, using the notation of Figures 1 and 2. It shows the allocations that each player can afford given his budget when prices are $p = v_1$ (i.e., according to player 1’s valuation). Allocations in the yellow rectangle (at or to the left of b_1) have value at most b_1 for player 1, and thus also price at most b_1 , so player 1 can afford them. Allocations in the red rectangle (at or to the right of b_1) have value at least b_1 for player 1, and thus player 1 values player 2’s allocation at most at $1 - b_1 = b_2$ (by normalization), so the price is at most b_2 and affordable for player 2. This blue striped area marks allocation with value for player 1 that is above his value for \mathcal{B} , and value for player 2 that is above his value for \mathcal{A} . The area has no allocation at all, as it is subset of the union of the following areas: the blue areas from Figure 1 without allocations below \mathcal{B} and to the left of \mathcal{A} ; the interiors of X and Y that are empty by the proof of Lemma 4.9 in Figure 2; and the interior of Z that must be empty, as an allocation there must be dominated by some PO allocation in the areas we just argued are empty, or by an anti-proportional PO allocation (which does not exist by assumption). Therefore, at these prices, if rectangle T_1 is empty then player 1 demands the allocation $\mathcal{B} = \check{S}^2$ (the rightmost allocation within the yellow area – his budget), while player 2 demands the allocation $\mathcal{A} = \check{S}^1$ (the highest allocation within the red area – his budget).

C.3 Proof of Theorem 4.10.

PROOF OF THEOREM 4.10. Let i be the player for which the conditions of Theorem 4.10 hold. By Lemma 4.9, both PO allocations \check{S}^1 and \check{S}^2 give both players their truncated share. To prove the theorem it is thus sufficient to show that at least one of these allocations is supported in a CE. We next show that indeed, for some $\gamma \in (0, 1)$, at least one of these two allocations is supported by item prices of the form $p_j = \gamma v_i(\{j\})$ for every item j .

We first characterize the set of allocations that are within the budget of each player when prices are set to $p_j = \gamma v_i(\{j\})$ for every item j , and $\gamma \in (0, 1)$ (i.e., prices are a linearly scaled down version of player i ’s valuation). Player i can afford any allocation \mathcal{S} such that $\gamma v_i(\mathcal{S}_i) \leq b_i$. Player k can afford any allocation \mathcal{S} such that $\gamma v_i(\mathcal{S}_k) \leq b_k$, or equivalently $\gamma(1 - v_i(\mathcal{S}_i)) \leq 1 - b_i$ (using that both valuations and budgets are normalized, that is, $b_1 + b_2 = v_1(M) = v_2(M)$). We illustrate this for $\gamma = 1$ in Figure 3.

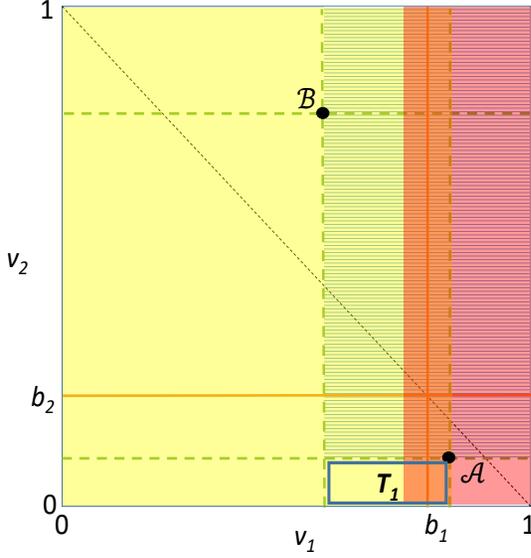


Fig. 4. This figure illustrates Case 1 in the proof of Theorem 4.10, using the notation of Figures 1 and 2, for player $i = 1$. Prices are $p_j = \gamma_1 v_1(\{j\})$ for every item j , where $\gamma_1 = b_1/b_1^+$. The yellow and red rectangles are the allocations that players 1 and 2 can afford, respectively. Both players can afford more allocations than when $\gamma = 1$ (cf. Figure 3). The overlap (the closure of the orange rectangle) contains the allocations that both players can afford. The value of γ_1 is such that player 1 can exactly afford allocation \mathcal{A} , which is clearly demanded by him at these prices (the rightmost allocation within his budget). We show in the proof that player 2 cannot yet afford allocation \mathcal{B} and any other allocation that gives him the same value, and so allocation \mathcal{A} is in his demand (the highest allocation within his budget, using that the blue striped area above \mathcal{A} that is within his budget, is empty). Thus (\mathcal{A}, p) is a CE.

Now define

$$\gamma_i = \max \left\{ \frac{b_i}{v_i(\check{\mathcal{S}}_i^i)}, \frac{1 - b_i}{1 - v_i(\check{\mathcal{S}}_i^k)} \right\} = \max \left\{ \frac{b_i}{b_i^+}, \frac{b_k}{v_i(\check{\mathcal{S}}_k^k)} \right\} < 1,$$

and note that γ_i is well-defined and less than 1. The assumption that the pair of budgets does not belong to $R_i(v_1, v_2)$ implies that the maximum is obtained by only one of the terms. The proof follows by analyzing two cases, as illustrated in Figures 4 and 5, respectively:

- **Case 1:** $\gamma_i = b_i/b_i^+$. We show that $\check{\mathcal{S}}^i$ is supported by item prices $p_j = \gamma_i v_i(\{j\})$. For player i , every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ that he can afford satisfies $v_i(\mathcal{S}_i) \leq b_i/\gamma_i$, and this holds with equality for $\check{\mathcal{S}}_i^i$. For player k , every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ that he can afford satisfies $(b_i/b_i^+)v_i(\mathcal{S}_k) \leq 1 - b_i$. Since we are in the case that $b_i/b_i^+ > (1 - b_i)/(1 - v_i(\check{\mathcal{S}}_i^k))$, we derive:

$$\frac{b_i}{b_i^+}(1 - v_i(\mathcal{S}_i)) = \frac{b_i}{b_i^+}v_i(\mathcal{S}_k) \leq 1 - b_i < \frac{b_i}{b_i^+}(1 - v_i(\check{\mathcal{S}}_i^k)),$$

or equivalently $v_i(\mathcal{S}_i) > v_i(\check{\mathcal{S}}_i^k)$. We claim that player k 's most preferred allocation that satisfies this is $\check{\mathcal{S}}^i$: By Lemma 4.9 it holds that $\check{\mathcal{S}}^i = \hat{\mathcal{S}}^k$, i.e., $\check{\mathcal{S}}^i$ is also the PO allocation in which player k gets at most b_k while maximizing his share. Since there is no allocation \mathcal{S} in which $v_i(\mathcal{S}_i) > v_i(\check{\mathcal{S}}_i^k)$ and $v_k(\mathcal{S}_k) > v_k(\check{\mathcal{S}}_k^k)$, the claim follows.

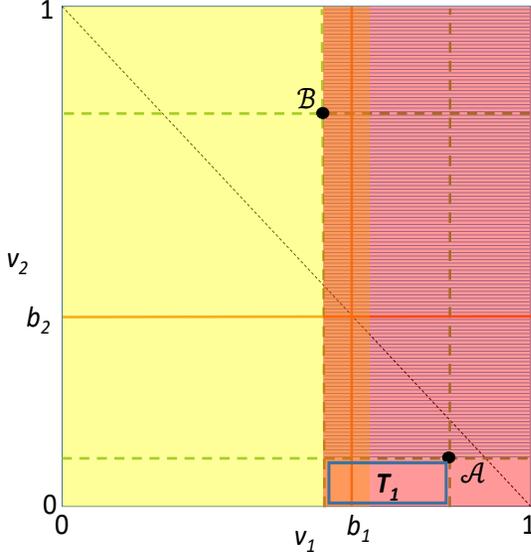


Fig. 5. This figure illustrates Case 2 in the proof of Theorem 4.10, using the notation of Figures 1 and 2, for player $i = 1$. Prices are $p_j = \gamma_1 v_1(\{j\})$ for every item j where $\gamma_1 = b_2/v_1(\check{S}_2^2)$. The yellow and red rectangles are as in Figure 4. The value of γ_1 is such that player 2 can exactly afford allocation \mathcal{B} , which is clearly demanded by him at these prices (the highest allocation within his budget, using that the blue striped area is empty and that \mathcal{B} is PO). We show in the proof that player 1 cannot yet afford allocation \mathcal{A} , and so allocation \mathcal{B} is in his demand (the rightmost allocation within his budget, using the fact that within his budget, there are no allocations that are also in the blue striped area right of \mathcal{B} or in T_1). Thus (\mathcal{B}, p) is a CE.

- **Case 2:** $\gamma_i = b_k/v_i(\check{S}_k^k)$. We show that \check{S}^k is supported by item prices $p_j = \gamma_i v_i(\{j\})$. For player k , every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ that he can afford satisfies $v_i(\mathcal{S}_k) \leq b_k/\gamma_i$, and this holds as equality for \check{S}_k^k . For player i , every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ that he can afford satisfies $v_i(\mathcal{S}_i) \leq b_i/\gamma_i < \frac{b_i}{b_i/v_i(\check{S}_i^i)} = v_i(\check{S}_i^i)$, as $b_i/v_i(\check{S}_i^i) < \frac{1-b_i}{1-v_i(\check{S}_i^i)} = \gamma_i$. Since T_i is empty, it cannot be the case that $v_i(\check{S}_i^i) < v_i(\mathcal{S}_i) < v_i(\check{S}_i^i)$. Thus the most preferred allocation that player i can afford gives him at most $v_i(\check{S}_i^i)$. This is indeed what he gets in allocation \check{S}^k , thus \check{S}_i^k is demanded by player i .

□

C.4 Additional missing proofs from Section 4.

PROOF OF PROPOSITION 4.5. Let v_1, v_2 be the (normalized) valuations and b_1, b_2 be the (normalized) budgets. Consider any budget-proportional PO allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$. Since $v_1(\mathcal{S}_1) \geq b_1$ and $v_2(\mathcal{S}_2) \geq b_2$, we have that $v_1(\mathcal{S}_2) = 1 - v_1(\mathcal{S}_1) \leq 1 - b_1 = b_2$ and $v_2(\mathcal{S}_1) = 1 - v_2(\mathcal{S}_2) \leq 1 - b_2 = b_1$.

We now construct a budget-exhausting combination pricing with parameters α, β . If it is the case that $v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) = 1$ (and thus $b_1 = v_1(\mathcal{S}_1)$ and $b_2 = v_2(\mathcal{S}_2)$), we set $\alpha = 1$ and $\beta = 0$. Otherwise, $v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1 \neq 0$, and we can set

$$\alpha = \frac{v_2(\mathcal{S}_2) - b_2}{v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1}, \beta = 1 - \alpha = \frac{v_1(\mathcal{S}_1) - b_1}{v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1}. \quad (5)$$

Observe that

$$\alpha v_1(\mathcal{S}_1) + \beta v_2(\mathcal{S}_1) = \frac{v_2(\mathcal{S}_2) - b_2}{v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1} v_1(\mathcal{S}_1) + \frac{v_1(\mathcal{S}_1) - b_1}{v_1(\mathcal{S}_1) + v_2(\mathcal{S}_2) - 1} (1 - v_2(\mathcal{S}_2)) = b_1,$$

and that similarly $\alpha v_1(\mathcal{S}_2) + \beta v_2(\mathcal{S}_2) = b_2$. Since the allocation is budget-proportional, it holds that $\alpha, \beta \geq 0$. In each of the two cases we thus have a combination pricing p with parameters α, β such that $p(\mathcal{S}_1) = b_1$ and $p(\mathcal{S}_2) = b_2$, and thus by Lemma 4.3, (\mathcal{S}, p) is a CE.

It remains to consider anti-proportional PO allocations. For every such allocation in which one player gets at most his budget-proportional share and the other gets less than his share, the same pair of parameters α, β defined in Equation (5) will give a budget-exhausting combination pricing with non-negative parameters, and thus a CE. \square

Example C.4. Consider 2 players with symmetric additive preferences who both value items (A, B, C, D) at $(7.9, 1, 5, 2)$. Player 1 has budget $b_1 = 1/2 + \epsilon$ and player 2 has budget $b_2 = 1/2 - \epsilon$ for some sufficiently small ϵ . Since the preferences are symmetric, every allocation is Pareto optimal, in particular the allocation $(\{B, C, D\}, \{A\})$ where player 2 gets a share of $\frac{7.9}{15.9} < b_2$. This allocation together with prices $p = (\frac{1}{2} - \epsilon, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} + \epsilon)$ for items (A, B, C, D) is a CE. However, the allocation $(\{A, B\}, \{C, D\})$ is also an equilibrium allocation, despite the fact that player 2's share drops to $\frac{7}{15.9}$. Corresponding equilibrium prices are $p = (\frac{1+\epsilon}{2}, \frac{\epsilon}{2}, \frac{1}{4}, \frac{1}{4} - \epsilon)$.

PROOF OF PROPOSITION 4.11. If there exists a budget-proportional or anti-proportional PO allocation, there exists a CE by Proposition 4.5. Otherwise the conditions of Theorem 4.10 hold for both players: any allocation in T_1 is dominated by $\hat{\mathcal{S}}^2$, and any allocation in T_2 is dominated by $\hat{\mathcal{S}}^1$, but there are no Pareto dominated allocations, so these two sets must be empty. By Theorem 4.10 there exists a CE in which every player gets at least his truncated share. \square

PROOF OF PROPOSITION 4.13 (SKETCH). Assume first there is an allocation that gives each player a value of exactly $1/2$. Then there is a PO allocation \mathcal{S} that gives each player at least $1/2$. Such an allocation is budget-proportional for $b_1 = b_2 = 1/2$, and thus by Proposition 4.5, there exists a CE (\mathcal{S}, p) . For sufficiently small $\epsilon > 0$, let $b_1 = (\frac{1}{2} + \epsilon)/(1 + \epsilon) > \frac{1}{2}$ and $b_2 = 1 - b_1$ (that is, we slightly increase the budget of player 1 while normalizing the sum $b_1 + b_2$ to 1). We claim that (\mathcal{S}, p) is also a CE with these budgets. Indeed, as prices have not changed, player 2 gets his demand. As for player 1, while his budget is slightly larger, he cannot afford any set that is more expensive than his set \mathcal{S}_1 , provided ϵ is smaller than the difference in prices of any two bundles with non-identical prices.

Consider now the complementary case, in which no allocation gives each player a value of exactly $1/2$. If there is an allocation that gives both players strictly more than $1/2$, consider any PO allocation that dominates it. For sufficiently small ϵ , the PO allocation is budget-proportional for any budgets $b_1 > b_2 \geq b_1 - \epsilon$, and so the result follows from Proposition 4.5. Note that if there is an allocation that gives both players strictly less than $1/2$ then the allocation in which the two players swap their bundles gives both players more than $1/2$. So from now on we assume that every allocation gives strictly more than $1/2$ to one player, and strictly less than $1/2$ to the other player. For sufficiently small ϵ , such an allocation is neither budget-proportional nor anti-proportional for any budgets $b_1 > b_2 \geq b_1 - \epsilon$.

Recall from Definition 4.7 that $\hat{\mathcal{S}}^1(b_1), \hat{\mathcal{S}}^2(b_2)$ are PO allocations that give players 1, 2 their truncated shares. As the set of PO allocations is finite, we can find $\epsilon > 0$ such that there is no PO allocation \mathcal{S} such that $1/2 - 2\epsilon < v_1(\mathcal{S}_1) < 1/2 + 2\epsilon$. For such an ϵ , consider budgets $b_1 > b_2 \geq b_1 - \epsilon$. Using the notation of Figure 2, let $\mathcal{A} = \hat{\mathcal{S}}^2(b_2)$ and $\mathcal{B} = \hat{\mathcal{S}}^1(b_1)$.

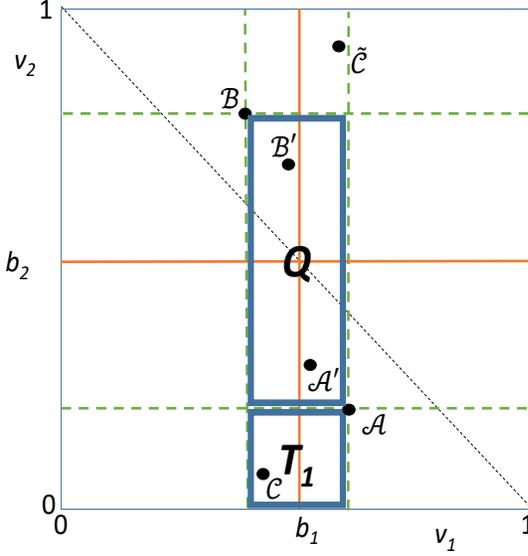


Fig. 6. This figure proves the main claims in Proposition 4.13, using the notation of Figures 1 and 2. The budgets b_1, b_2 are almost equal, i.e., both are very close to $1/2$. For every allocation \mathcal{S} there is a “symmetric” allocation $\tilde{\mathcal{S}}$ obtained by swapping the allocated bundles, which corresponds to a 180° rotation around the point $(1/2, 1/2)$. Assume for contradiction that \mathcal{A} is not symmetric to \mathcal{B} , that is $\mathcal{B} \neq \tilde{\mathcal{A}} = \mathcal{B}'$. Then it is also the case that $\mathcal{A} \neq \tilde{\mathcal{B}} = \mathcal{A}'$. Notice that for sufficiently small ϵ , one of $\mathcal{A}', \mathcal{B}'$ must be located in the interior of the axes-parallel rectangle Q as illustrated in the figure. But since $\mathcal{A} = \hat{\mathcal{S}}^2(b_2)$ gives player 2 his truncated share closest to $b_2 \approx 1/2$ from below, $\mathcal{A}' \neq \mathcal{A}$ cannot be located as in the figure. Similarly, $\mathcal{B}' \neq \mathcal{B}$ cannot be located in the interior of Q as in the figure, a contradiction. We have thus established that \mathcal{A}, \mathcal{B} are symmetric. We now show that the closure of T_1 must be empty, except for \mathcal{A} : If that were not the case – say, T_1 contained an allocation $\mathcal{C} \neq \mathcal{A}$ – then its symmetric allocation $\tilde{\mathcal{C}}$ would Pareto dominate \mathcal{B} (due to the symmetry of \mathcal{A}, \mathcal{B}), a contradiction. T_2 only contains the allocation \mathcal{B} by a similar argument using the Pareto optimality of \mathcal{A} .

We first claim that $\mathcal{A} = \hat{\mathcal{S}}^2(1/2)$ and $\mathcal{B} = \hat{\mathcal{S}}^1(1/2)$. This holds because \mathcal{B} is the PO allocation that gives the largest value to player 1 that is below $1/2 - 2\epsilon$, but there are no PO allocations that give player 1 value between $1/2 - 2\epsilon$ and $1/2$, and thus it also gives the largest value to player 1 that is below $1/2$. A similar argument holds for \mathcal{A} and the truncated share of player 2.

Because every allocation gives strictly more than $1/2$ to one player and strictly less to the other, $v_2(\mathcal{A}_2) < 1/2 \implies v_1(\mathcal{A}_1) > 1/2$, and $v_1(\mathcal{B}_1) < 1/2 \implies v_2(\mathcal{B}_2) > 1/2$.

We now show that \mathcal{A} and \mathcal{B} are “symmetric” in the sense that one is obtained from the other by swapping bundles among the players (and so $v_1(\mathcal{A}_1) = v_1(\mathcal{B}_2) = 1 - v_1(\mathcal{B}_1)$ and $v_2(\mathcal{A}_2) = 1 - v_2(\mathcal{B}_2)$, as can be seen in Figure 6). The proof of the symmetry claim appears in Figure 6, which also shows that T_1, T_2 as defined for Theorem 4.10 are both empty. We can set ϵ to be sufficiently small such that $(b_1, b_2) \notin R_i(v_1, v_2)$ (such ϵ exists as $R_i(v_1, v_2)$ is finite). The proof is complete by invoking Theorem 4.10. \square

D COMPUTATIONAL ASPECTS.

Computational problems associated with markets with discrete goods tend to be computationally hard. Let us list the most relevant ones in our model, together with observations about their computational complexity. Throughout we assume the following:

- Players have strict additive preferences (see Remark D.1).
- There are m items and n players; the input and output sizes are polynomial in m and n .
- An additive preference is given by a vector of positive rational numbers $v = (v^1, v^2, \dots, v^m)$.
- A budget is given by a positive rational number $b > 0$.
- A set of items S is given by its characteristic vector.
- An allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n)$ is given by listing the sets in it.
- A price vector is given as a list of positive rational numbers $p = (p_1, p_2, \dots, p_m)$.
- A perturbation is given by a positive rational number $\epsilon > 0$.

We next list the problems and their status.

Problem 1: Is a given set a demand set? **Input:** A preference v , a budget b , prices p , a set S . **Question:** Is S the demanded set of the given preference and budget under these prices. **Status:** co-NP-complete (even for $n = 2$) by an easy reduction from Knapsack.

Problem 2: Demand Oracle. **Input:** A preference v , a budget b , prices p . **Output:** Find S that is the demanded set of the given preference and budget under these prices. **Status:** NP-complete under Turing reductions (even for $n = 2$). (The status is stated for Turing reductions since this is not a decision problem; formally, NP-completeness applies to the corresponding decision problem.) Hardness can be seen e.g. from the hardness of Problem 4 below.

Problem 3: Is a given allocation PO? **Input:** Preferences v_1, v_2, \dots, v_n , and an allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n)$. **Question:** Is the given allocation PO for these preferences? **Status:** co-NP-complete even for $n = 2$ (this was shown for larger n in [25]). We can reduce from the Partition problem: We are given a list of positive integer numbers that sum to K and asked whether they can be partitioned into two sets each whose sum is exactly $K/2$. We treat each of these numbers as an item where both players have this value for the item (we call these “regular” items), and add two new items whose values for the first player are $K/2 - \epsilon$ and $K/2 + 2\epsilon$, respectively, while for the second player they are the opposite, $K/2 + 2\epsilon$ and $K/2 - \epsilon$, respectively (ϵ is chosen to be sufficiently small). Consider the allocation giving all the regular items to the first player and the two new items to the second player. It is Pareto-optimal if and only if there is no way in which the regular items can be split into two halves with the same sums (in which case each player would get one of these halves as well as the new item that he likes).

Problem 4: Is a pair of given allocation and prices a CE? **Input:** Preferences v_1, v_2, \dots, v_n , budgets b_1, b_2, \dots, b_n , an allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n)$, and prices p . **Question:** Do the given allocation and price vector form a CE for these preferences? **Status:** co-NP-complete even for $n = 2$. Consider the construction of Problem 3, and add to it budgets of $K + 2\epsilon$ for the first player and $K + 4\epsilon$ for the second player, and prices where each regular item’s price is exactly its value and the new items are priced at $K/2 + 2\epsilon$ each. Then this is an equilibrium if and only if there is no partition into two equal halves.

Problem 5: Can a given allocation be a CE allocation? **Input:** Preferences v_1, v_2, \dots, v_n , budgets b_1, b_2, \dots, b_n and an allocation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n)$. **Question:** Are there prices p that with the given allocation give a CE for these preferences? **Status:** co-NP-complete, even for $n = 2$. Note that it is co-NP-complete and *not* NP-complete, despite what the syntactic form may suggest.

To show that it is in co-NP notice that the existence of equilibrium prices for the given allocation can be captured by a linear program whose inequalities are: (a) S_i is in budget for player i ; (b) every set T that is preferred by i to S_i is out of budget for i . While there are exponentially many such inequalities, if the LP is infeasible then this is demonstrated already by $n + 1$ inequalities whose specification is the required proof. Hardness follows from the construction in Problem 4, which provides prices for the Pareto-optimal allocation used in the reduction of Problem 3, while certainly no such prices exist otherwise.

Problem 6: Do given preferences have a CE? **Input:** Preferences v_1, v_2, \dots, v_n and budgets b_1, b_2, \dots, b_n . **Question:** Is there a CE for these preferences? **Status:** NP-hard even for $n = 2$ and $b_1 = b_2 = 1/2$ by a reduction similar to that in Problem 3 (see [26], Remark D.1 and Appendix D.1).

Problem 7: Do given preferences with generic budgets have a CE? **Input:** Preferences v_1, v_2, \dots, v_n , budgets b_1, b_2, \dots, b_n , perturbation ϵ . **Question:** Is there a CE for these preferences with budgets perturbed by at most ϵ ? **Status:** Open. In particular, it may be the case that the answer is always “yes” (or at least that it is always “yes” for $n = 2$).

REMARK D.1. *In the above reductions from hard problems to Fisher markets, the resulting preferences can be non-strict. However we can add small perturbations to transform them into strict preferences without affecting the arguments. Consider for example the reduction from the Partition problem to Problem 6 for Fisher markets with two players and $b_1 = b_2 = 1/2$. Let v be the preference for the items resulting from letting the value of each item be the corresponding integer. Add small perturbations that result in two different preferences v_1, v_2 for the items as follows: For every two bundles $S \neq T$ such that $v(S) = v(T)$, $v_1(S) > v_1(T)$ and $v_2(T) > v_2(S)$, and for every two bundles $S' \neq T'$ such that $v(S') > v(T')$, $v_1(S') > v_1(T')$ and $v_2(S') > v_2(T')$ (this can be achieved e.g. by small perturbations that result in two opposite lexicographic tie-breaking rules). It is not hard to verify that the reduction still holds, i.e., there is a CE in the market (with prices equal to the unperturbed integers) if and only if there is a partition into equal halves.*

D.1 The NP-hardness result of Deng et al. [26].

We briefly review the NP-hardness result of [26] for Problem 6 above, noting the differences from our model. The NP-hardness result has two parts: For 2 players, the reduction is from the Partition problem (as sketched in Remark D.1), and for n players, the reduction is from the Exact Cover by 3-Sets problem (and establishes hardness of determining existence of even an *approximate* CE). One superficial difference is that the result is stated for Arrow-Debreu exchange economies rather than for Fisher markets, however it immediately applies to Fisher markets (as noted also in [50]) by simply removing the sellers (for this reason we say the reduction from Partition results in 2 rather than 3 players – we are counting only the buyers). Another superficial difference is non-strictness of the preferences (see Remark D.1). The main difference is non-genericity of the budgets, due to which the NP-hardness result is not directly applicable to our model. For 2 players we have already noted that, in the Fisher market resulting from the reduction, any perturbation of the equal budgets leads to equilibrium existence. We do not know whether this is also the case for n players, and leave it as an open question.

E COMPUTERIZED SEARCH FOR EQUILIBRIA.

We attempted to find CEs, or find a market with no CEs, for both additive and general preferences. The instances examined were either randomly generated by sampling from distributions, or taken from real-world Spliddit data.

E.1 2 players with randomly sampled preferences.

Setup. Our computerized search ran on instances with between 4 and 8 items and 2 players with randomly generated preferences.¹⁸ We generated both random additive preferences, where the values for the items were drawn from the uniform or Pareto distributions and then normalized to sum up to 1, as well as random general monotone preferences. To generate the monotone preferences we randomly picked an order for all singletons, then randomly placed all pairs among the singletons while maintaining monotonicity, then placed all triplets and so on.

In choosing budgets for the random additive instances, our goal was to avoid instances for which we know from Proposition 4.5 or Theorem 4.10 that a CE exists. We thus iterated over consecutive pairs of allocations on the Pareto optimal frontier, and for each such pair tested several budgets that “crossed” in between those allocations. To illustrate this, recall Figure 2 in which the budgets “cross” between A and B . This choice ruled out the existence of budget-proportional allocations. We used additional such considerations to carefully chose the budgets in order to rule out all “easy cases”. For random non-additive instances, we simply used several choices of arbitrary non-equal budgets.

Running the search. For each of the resulting instances we conducted an exhaustive search for an equilibrium: we iterated over all possible PO allocations, and for each one of them we used CVX with the LP solver MOSEK 7 to look for equilibrium prices (see the linear program described in Appendix D, Problem 5). Note that although the problem is possibly computationally hard, our instances were small enough that they could be completely solved by the solver in a matter of seconds. We verified the equilibria found by the LP solver by implementing a demand oracle. The run time for 10,000 instances of 4 items was several minutes, and run time increased noticeably as the number of items increased.

In all instances with additive preferences that we tested, we found and verified an equilibrium. As for general preferences, in all cases with 4 items we found an equilibrium (as expected), and even for 5 items we needed to go over several hundred instances before we found one that does not have an equilibrium. Instances with general preferences that do not have an equilibrium seemed to become more rare as the number of items increased.

E.2 Spliddit data with additive preferences.

Setup. We ran our second computerized search on instances of Spliddit data, specifically, 803 instances created so far through Spliddit’s “divide goods” application that were kindly provided to us by the Spliddit team [cf. 20, Sec. 4.3]. In every Spliddit instance, every player divides a pool of 1000 points among the instance’s indivisible items in order to indicate his values for the items; the resulting preference is additive in these values.

Running the search. We implemented a simple tâtonnement process: Prices start at 0, and all players are asked for their demand at these prices. Then the price of over-demanded items is increased by 1, and the price of undemanded items is decreased by 1. Prices thus remain integral throughout the process, and since our budgets are reasonably-sized integers the process is likely to converge reasonably quickly (we do not allow it to run for more than 20,000 iterations). The running time was typically well under a minute, usually no more than a second or two. One issue that deserves mention (and possibly further research) is how to update the prices when more than a single item is over- or under-demanded. Our first attempts either updated only a single such item’s price in every iteration, or updated all such items’ prices – both variants converged to a CE

¹⁸The instances with 4 items were generated as a “sanity check”, as we know from Proposition C.1 that a CE exists for these instances.

fairly often. We improved upon this by randomly deciding after each price update whether or not to continue updating prices in the current iteration.

Special case of interest. An anecdotal but interesting case is non-demo instances with between 5 and 10 items. There were 14 such instances available in the data, with between 3 and 9 players each. As Spliddit assumes that players have equal entitlements, we started by giving all players equal budgets of 100, in which case an equilibrium was found for less than half of the instances. When we added small perturbations to make the budgets only *almost* equal (resulting in the budget vector $(100, 103, 106, \dots)$), we found a CE in all instances. The same was true for other small perturbations that we tried (resulting in budget vectors like $(100, 101, 104, 109, \dots)$). We also tried several other budget vectors with budgets that are far from equal (such as $(100, 151, 202, \dots)$ or $(100, 200, 300, \dots)$), and CEs were always found for these as well.